Max-plus algebra

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Outline

1 Max-plus algebra
   - Discrete Dynamic Systems
   - Max-plus algebra
   - Operations with vectors and matrices
   - Computational complexity
   - Finiteness
Systems

1. **continuous**
   - the state varies continuously through time
   - outputs in form of continuous functions
     \[ x = x(t) \quad t \text{ continuous variable representing time} \]

2. **discrete (DDS)**
   - the state varies discretely, i.e. from event to event
   - outputs in form of discrete functions
     \[ x = x(r) \quad r \text{ discrete variable representing event} \]
Illustration of Discrete Dynamic Systems

Examples of DDS

- traffic systems (bus services, ...)
- digital signal processing
- industrial production
Description of the system:

The production line consists of several machines. The work of any individual machine can be influenced by the work of the others. Thus the system is given by two classes of parameters. On one hand the starting time of each machine must be given, on the other hand it must be given whether the individual machines wait (and how long) before proceeding to next event until certain others have completed their current events.
Model system - production line - illustration

work for 1. machine
- working time of the 1. machine is 3 time units, i.e. waits for itself 3 time units
- 2. machine 4 time units
- 4. machine 8 time units

$x_i(r)$ completion of the $r$th work-piece on the $i$th machine

$$x_1(r + 1) = \max\{x_1(r) + 3, x_2(r) + 4, x_4(r) + 8\}$$
Questions

- What is the maximum speed at which the system can run?
- What is the starting time of individual machines to achieve the steady-state, i.e. no traffic jam appears at any machine?
- Knowing the time of delivery, what is the latest time at which the production should begin?
Aim:

- to formalize the notation of arbitrary DDS by vectors and matrices,
- to define "proper algebra", in which are the classical operations (addition and multiplication) replaced by pair of more "proper operations"
- to look for answers for above questions
Formalization of notation - introducing element $\varepsilon$

$x_1(r + 1) = \max\{x_1(r) + 3, x_2(r) + 4, x_4(r) + 8\}$

formally

$x_1(r + 1) = \max\{x_1(r) + a_{11}, x_2(r) + a_{12}, x_4(r) + a_{14}\}$

$$\max\{x_1(r) + a_{11}, x_2(r) + a_{12}, x_3(r) + a_{13}, x_4(r) + a_{14}\} =$$

let $a_{13} = -\infty$ $\implies$

$$= \max\{x_1(r) + a_{11}, x_2(r) + a_{12}, x_3(r) + (\infty), x_4(r) + a_{14}\} =$$

$$= \max\{x_1(r) + a_{11}, x_2(r) + a_{12}, -\infty, x_4(r) + a_{14}\} =$$

$$= \max\{x_1(r) + a_{11}, x_2(r) + a_{12}, x_4(r) + a_{14}\} =$$

$$= x_1(r + 1)$$

Notation: $-\infty = \varepsilon$
Formalization of notation - DDS given by forward recursion

System with $n$ machines described by forward recursion:

$$x_i(r + 1) = \max \{ x_1(r) + a_{i1}, x_2(r) + a_{i2}, \ldots, x_n(r) + a_{in} \}$$

- $i$th machine $i = 1, 2, \ldots, n$
- $(r + 1)$th event $r = 1, 2, \ldots$
Formalization of notation - model system with 4 machines

\[ x_1(r + 1) = \max\{x_1(r) + a_{11}, x_2(r) + a_{12}, x_3(r) + a_{13}, x_4(r) + a_{14}\} \]

\[ x_2(r + 1) = \max\{x_1(r) + a_{21}, x_2(r) + a_{22}, x_3(r) + a_{23}, x_4(r) + a_{24}\} \]

\[ x_3(r + 1) = \max\{x_1(r) + a_{31}, x_2(r) + a_{32}, x_3(r) + a_{33}, x_4(r) + a_{34}\} \]

\[ x_4(r + 1) = \max\{x_1(r) + a_{41}, x_2(r) + a_{42}, x_3(r) + a_{43}, x_4(r) + a_{44}\} \]
Definition

Let $A = (a_{ij})$ be a square matrix of order $n$. $A$ is the so-called transition matrix of a discrete dynamic system with $n$ machines, in which the element $a_{ij}$ is a finite real number or equal to $\varepsilon$ and represents the influence of $j$th machine on $i$th machine.

We can take the following matrix for the model system introduced partially in the beginning:

$$A = \begin{pmatrix}
3 & 4 & \varepsilon & 8 \\
2 & 5 & 2 & \varepsilon \\
4 & 6 & 3 & 4 \\
\varepsilon & 3 & 0 & 6
\end{pmatrix}$$
Definition

Let $\mathbb{R}$ be the set of real numbers. Let $\mathbb{R}^* = \mathbb{R} \cup \{-\infty\}$. Let $a \oplus b = \max\{a, b\}$ and $a \otimes b = a + b$, for $a, b \in \mathbb{R}^*$. The ordered triple $(\mathbb{R}^*, \oplus, \otimes)$ is the so-called max-plus algebra.

Example:

1. $3 \otimes 4 = 3 + 4 = 7$
2. $4 \otimes 3 = 4 + 3 = 7$
3. $5 \oplus 8 = \max\{5, 8\} = 8$
4. $8 \oplus 5 = \max\{8, 5\} = 8$
5. $2 \otimes (4 \oplus 6) = 2 + \max\{4, 6\} = 2 + 6 = 8$
6. $2 \otimes 4 \oplus 2 \otimes 6 = \max\{2 + 4, 2 + 6\} = \max\{6, 8\} = 8$
Properties of operations $\otimes$ and $\oplus$ for scalars $I$.

**Theorem**

The scalars $a, b, c \in \mathbb{R}^*$ satisfy the following rules:

1. **Commutative law $\oplus$**
   \[ a \oplus b = b \oplus a \]

2. **Associative law $\oplus$**
   \[ a \oplus (b \oplus c) = (a \oplus b) \oplus c \]

3. **Commutative law $\otimes$**
   \[ a \otimes b = b \otimes a \]

4. **Associative law $\otimes$**
   \[ a \otimes (b \otimes c) = (a \otimes b) \otimes c \]

5. **Distributive law for $\otimes$ over $\oplus$**
   \[ a \otimes (b \oplus c) = a \otimes b \oplus a \otimes c \]
Properties of operations $\otimes$ and $\oplus$ for scalars II.

**Theorem**

For $a \in \mathbb{R}^*$ the following hold

\[
a \oplus \varepsilon = \varepsilon \oplus a = a \quad \text{\varepsilon neutral element over } \oplus
\]

\[
a \otimes \varepsilon = \varepsilon \otimes a = \varepsilon \quad \text{\varepsilon absorbing over } \otimes
\]

\[
a \otimes 0 = 0 \otimes a = a \quad 0 \text{ neutral element over } \otimes
\]

**Example:**

1. \[3 \oplus \varepsilon = \varepsilon \oplus 3 = \max\{3, \varepsilon\} = 3\]
2. \[4 \otimes \varepsilon = \varepsilon \otimes 4 = \varepsilon + 4 = \varepsilon\]
3. \[5 \otimes 0 = 0 \otimes 5 = 0 + 5 = 5\]
Description of the system with $n$ machines by forward recursion

- In conventional algebra

$$x_i(r + 1) = \max\{x_1(r) + a_{i1}, x_2(r) + a_{i2}, \ldots, x_n(r) + a_{in}\}$$

- In max-plus algebra

$$x_i(r + 1) = a_{i1} \otimes x_1(r) \oplus a_{i2} \otimes x_2(r) \oplus \ldots \oplus a_{in} \otimes x_n(r)$$
Max-plus algebra

**Definition of the power**

**Definition**

Let \( a \in \mathbb{R}^* \). Let \( p \in \mathbb{N}^+ \). \( p \text{th power of } a \) is the number

\[
a^p = a \boxdot a \boxdot \cdots \boxdot a
\]

\( p \) times

\( a^0 = 0 \)

Remark: \( a^p = p \times a \)

**Example:**

1. \( 3^5 = 3 \boxdot 3 \boxdot 3 \boxdot 3 \boxdot 3 = 5 \times 3 = 15 \)

2. \( \varepsilon^2 = \varepsilon \boxdot \varepsilon = 2 \times \varepsilon = \varepsilon \)
Theorem (Binomial theorem)

Let $a, b \in \mathbb{R}^*$. Let $p \in \mathbb{N}$. The following holds

$$(a \oplus b)^p = a^p \oplus b^p$$

Example:

$$(4 \oplus 5)^3 = 3 \times (4 \oplus 5) = 3 \times 4 \oplus 3 \times 5 = 4^3 \oplus 5^3$$
The above principle holds for arbitrary number of summands

**Theorem (Principle of exponentiation)**

Let $a_i \in \mathbb{R}^*$, for $i = 1, 2, \ldots, n$. The following holds

\[
\left( \sum_i \oplus a_i \right)^p = \sum_i \oplus (a_i)^p
\]
Differences from conventional algebra

Theorem

Let \( a, b \in \mathbb{R}^* \). The following holds

\[
a \oplus a = a \quad \text{Idempotent law of addition} \quad \oplus
\]

\[
a \oplus b \geq a \quad \text{Majority law of addition} \quad \oplus
\]

Example:

1. \( 7 \oplus 7 = \max\{7, 7\} = 7 \)
2. \( 7 \oplus (-4) = \max\{7, -4\} \geq 7 \)
The above principle holds for arbitrary number of summands

**Theorem (Principle of majority)**
Let \( a_i \in \mathbb{R}^* \), for \( i = 1, 2, \ldots, n \). The following holds

\[
\sum_{i}^{\oplus} a_i \geq a_i, \quad i = 1, 2, \ldots, n
\]

Sum (maximum) of arbitrary number of elements is greater or equal to each of the considered elements.
Max-plus algebra

**Structure** \((\mathbb{R}^*, \oplus, \otimes)\)

1. \((\mathbb{R}, +, (\otimes))\) is a linearly ordered \((\leq)\) commutative group
   - closed with respect to \(\otimes\)
   - associative
   - with neutral element 0
   - with inverse element of \(a\) (finite!) is \(-a\)
   - commutative

   the equation \(n \times x = b\) is solvable with solution \(x = \frac{b}{n}\).

2. \((\mathbb{R}^*, \text{max}(\oplus))\) satisfies
   - closed with respect to \(\oplus\)
   - associative
   - with neutral element \(\varepsilon\)
   - commutative

3. \(\varepsilon\) absorbing with respect to \(\otimes\)

4. distributivity of \(\otimes\) with respect to \(\oplus\)
Definition

Let $m, n \in \mathbb{N}$. The set of all $n$-dimensional column vectors over $\mathbb{R}^*$ will be denoted by $\mathbb{R}^*(n)$, the set of all $m \times n$ matrices will be denoted by $\mathbb{R}^*(m, n)$.

Operations with matrices are defined formally in the same manner as the matrix operations over conventional algebra with respect to $\oplus$ and $\otimes$. 
Operations with vectors and matrices

Multiplication of matrix by scalar

Definition (Multiplication of matrix by scalar)

Let $A \in \mathbb{R}^*(m, n)$ and $\alpha \in \mathbb{R}^*$. We define the $\alpha$-multiple of the matrix $A$ as the matrix $C \in \mathbb{R}^*(m, n)$, $C = (c_{ij})$, for which $c_{ij} = \alpha \otimes a_{ij}$, for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$.

Notation: $C = \alpha \otimes A$

Example:

$$A = \begin{pmatrix} 1 & 0 & 4 \\ 3 & 2 & 0 \\ 5 & 3 & \varepsilon \end{pmatrix} \quad \alpha = 2$$

$$\alpha \otimes A = 2 \otimes \begin{pmatrix} 1 & 0 & 4 \\ 3 & 2 & 0 \\ 5 & 3 & \varepsilon \end{pmatrix} = \begin{pmatrix} 2 + 1 & 2 + 0 & 2 + 4 \\ 2 + 3 & 2 + 2 & 2 + 0 \\ 2 + 5 & 2 + 3 & 2 + \varepsilon \end{pmatrix} = \begin{pmatrix} 3 & 2 & 6 \\ 5 & 4 & 2 \\ 7 & 5 & \varepsilon \end{pmatrix}$$
Addition of matrices

Definition

Let $A \in \mathbb{R}^*(m, n)$ and $B \in \mathbb{R}^*(m, n)$. We define the sum of matrices $A$ and $B$ as the matrix $C \in \mathbb{R}^*(m, n)$, $C = (c_{ij})$, for which $c_{ij} = a_{ij} \oplus b_{ij}$, for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$.

Notation: $C = A \oplus B$

Example:

\[
A = \begin{pmatrix}
1 & 0 & 4 \\
3 & 2 & 0 \\
5 & 3 & \varepsilon
\end{pmatrix}
\quad B = \begin{pmatrix}
0 & \varepsilon & 6 \\
-1 & 0 & 4 \\
1 & 2 & \varepsilon
\end{pmatrix}
\]

\[
A \oplus B = \begin{pmatrix}
\text{max}\{1, 0\} & \text{max}\{0, \varepsilon\} & \text{max}\{4, 6\} \\
\text{max}\{3, -1\} & \text{max}\{2, 0\} & \text{max}\{0, 4\} \\
\text{max}\{5, 1\} & \text{max}\{3, 2\} & \text{max}\{\varepsilon, \varepsilon\}
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 6 \\
3 & 2 & 4 \\
5 & 3 & \varepsilon
\end{pmatrix}
\]
### Properties of matrix addition

**Theorem**

Let $A \in \mathbb{R}^*(m, n)$, $B \in \mathbb{R}^*(m, n)$ and $C \in \mathbb{R}^*(m, n)$. The following holds:

\[
A \oplus B = B \oplus A \quad \text{Commutative law } \oplus
\]

\[
A \oplus (B \oplus C) = (A \oplus B) \oplus C \quad \text{Associative law } \oplus
\]

\[
\alpha \otimes (A \oplus B) = \alpha \otimes A \oplus \alpha \otimes B
\]

\[
\alpha \otimes (\beta \otimes A) = (\alpha \otimes \beta) \otimes A
\]
Multiplication of matrices

Definition

Let $A \in \mathbb{R}^*(m, r)$ and $B \in \mathbb{R}^*(r, n)$. We define the product of matrices $A$ and $B$ as the matrix $C \in \mathbb{R}^*(m, n)$, $C = (c_{ij})$, for which

$$c_{ij} = a_{i1} \otimes b_{1j} \oplus a_{i2} \otimes b_{2j} \oplus \cdots \oplus a_{ir} \otimes b_{rj} = \sum_k a_{ik} \otimes b_{kj},$$

for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$.

Notation: $C = A \otimes B$

Example:

$$A \otimes B = \begin{pmatrix} 2 & 0 & -1 \\ 3 & 2 & 0 \end{pmatrix} \otimes \begin{pmatrix} 5 & 4 & 1 \\ 1 & 3 & -1 \\ 0 & 2 & 3 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{pmatrix}$$

$$c_{12} = 2 \otimes 4 \oplus 0 \otimes 3 \oplus (-1) \otimes 2 = \max\{2 + 4, 0 + 3, -1 + 2\} = 6$$
Max-plus algebra

Operations with vectors and matrices

Application - product line

Description of a model system with 4 machines in max-plus algebra:

\[
\begin{align*}
    x_1(r + 1) &= a_{11} \otimes x_1(r) \oplus a_{12} \otimes x_2(r) \oplus a_{13} \otimes x_3(r) \oplus a_{14} \otimes x_4(r) \\
    x_2(r + 1) &= a_{21} \otimes x_1(r) \oplus a_{22} \otimes x_2(r) \oplus a_{23} \otimes x_3(r) \oplus a_{24} \otimes x_4(r) \\
    x_3(r + 1) &= a_{31} \otimes x_1(r) \oplus a_{32} \otimes x_2(r) \oplus a_{33} \otimes x_3(r) \oplus a_{34} \otimes x_4(r) \\
    x_4(r + 1) &= a_{41} \otimes x_1(r) \oplus a_{42} \otimes x_2(r) \oplus a_{43} \otimes x_3(r) \oplus a_{44} \otimes x_4(r)
\end{align*}
\]

Matrix notation: \( x(r + 1) = A \otimes x(r) \)
Remark:

Let \( C = A \otimes B \). Each column of the product matrix is the vector which will arise by applying the matrix \( A \) to the corresponding column of \( B \). This property will be used later on under name principle of column action.

Example:

\[
A \otimes B = \begin{pmatrix} 2 & 0 & -1 \\ 3 & 2 & 0 \end{pmatrix} \otimes \begin{pmatrix} 5 & 4 & 1 \\ 1 & 3 & -1 \\ 0 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 7 & 6 & 3 \\ 8 & 7 & 4 \end{pmatrix}
\]
Let $A$, $B$ and $C$ be matrices of proper types over $\mathbb{R}^*$. The following holds

\[
A \otimes (B \otimes C) = (A \otimes B) \otimes C \quad \text{Associative law } \otimes
\]

\[
A \otimes (B \oplus C) = A \otimes B \oplus A \otimes C \quad \text{Distributive law for } \otimes \text{ over } \oplus
\]

\[
\alpha \otimes (A \otimes B) = A \otimes (\alpha \otimes B)
\]

**Remark:**

The above rules hold as well as for column vectors (matrices with one column). In this case for an arbitrary matrix $A$ and a column vector $x$ has the last equation the form

\[
\alpha \otimes (A \otimes x) = A \otimes (\alpha \otimes x).
\]
Application - production line

Application:

Let us consider a production line in which a power cut occurs during the work. It imposes a time delay of length $\delta$ on all $r$th events, for some $r$. If $x_i(r)$ was the finishing time of $r$th event on the $i$th machine, then this value increases by $\delta$, for each $i$. Thus the new $(r + 1)$st event-times will be represented by the vector

$$
\begin{pmatrix}
  x_1(r) + \delta \\
  \vdots \\
  x_n(r) + \delta 
\end{pmatrix} = \delta \otimes x(r)
$$

Thus the $(r + 1)$st event as well as the following events will be affected by the same delay $\delta$

$$
x'(r + 1) = A \otimes (\delta \otimes x(r)) = \delta \otimes (A \otimes x(r)) = \delta \otimes x(r + 1).
$$
We define a **diagonal matrix** $D \in \mathbb{R}^*(n, n)$, $D = (d_{ij})$ as one in which for $i, j = 1, 2, \ldots, n$

$$d_{ij} = \begin{cases} 
\varepsilon & \text{for } i \neq j \\
\text{arbitrary} & \text{for } i = j
\end{cases}$$

Notation: $D = \text{diag}(d_{11}, d_{22}, \ldots, d_{nn})$

**Example:**

$$D = \begin{pmatrix} 
3 & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & 1
\end{pmatrix} = \text{diag}(3, \varepsilon, 1)$$
Definition

We define the identity matrix $E \in \mathbb{R}^*(n, n)$, $E = (e_{ij})$ as one in which for $i, j = 1, 2, \ldots, n$

$$e_{ij} = \begin{cases} 
\varepsilon & \text{for } i \neq j \\
0 & \text{for } i = j 
\end{cases}$$

Example:

$$E = \begin{pmatrix} 
0 & \varepsilon & \varepsilon \\
\varepsilon & 0 & \varepsilon \\
\varepsilon & \varepsilon & 0 
\end{pmatrix} = \text{diag}(0, 0, 0)$$
Theorem

Let $E \in \mathbb{R}^*(n, n)$ be the identity matrix. For arbitrary matrix $A \in \mathbb{R}^*(n, n)$ holds

$$A \otimes E = E \otimes A = A$$

Remark: The identity matrix $E$ is the neutral element with respect to matrix multiplication.

Example:

$$A \otimes E = \begin{pmatrix} 5 & 4 & 1 \\ 1 & 3 & -1 \\ 0 & 2 & 3 \end{pmatrix} \otimes \begin{pmatrix} 0 & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon \\ \varepsilon & \varepsilon & 0 \end{pmatrix} = \begin{pmatrix} 5 & 4 & 1 \\ 1 & 3 & -1 \\ 0 & 2 & 3 \end{pmatrix}$$
We define the null matrix $\Phi \in \mathbb{R}^*(n, n)$, $\Phi = (\varphi_{ij})$ as one in which $\varphi_{ij} = \varepsilon$ for $i, j = 1, 2, \ldots, n$.

Example:

$$\Phi = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon \end{pmatrix} = \text{diag}(\varepsilon, \varepsilon, \varepsilon)$$
Theorem

Let $\Phi \in \mathbb{R}^*(n, n)$ be the null matrix. For arbitrary matrix $A \in \mathbb{R}^*(n, n)$ holds

$$A \oplus \Phi = \Phi \oplus A = A$$

Remark: The null matrix $\Phi$ is the neutral element with respect to matrix addition.

Example:

$$A \oplus \Phi = \begin{pmatrix} 5 & 4 & 1 \\ 1 & 3 & -1 \\ 0 & 2 & 3 \end{pmatrix} \oplus \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon \end{pmatrix} = \begin{pmatrix} 5 & 4 & 1 \\ 1 & 3 & -1 \\ 0 & 2 & 3 \end{pmatrix}$$
Matrix powering

**Definition**

Let $A \in \mathbb{R}^*(n, n)$. Let $r \in \mathbb{N}$. We define the $r$th power of matrix $A$, $A^r$, as follows:

$$A^r = \begin{cases} 
E & r = 0 \\
A \otimes A^{r-1} & r = 1, 2, \ldots 
\end{cases}$$

Notation: $A^r = (a_{ij}^{(r)})$

**Example:**

$$\begin{pmatrix} 2 & -3 \\ \varepsilon & 4 \end{pmatrix}^3 = \begin{pmatrix} 2 & -3 \\ \varepsilon & 4 \end{pmatrix} \otimes \begin{pmatrix} 2 & -3 \\ \varepsilon & 4 \end{pmatrix}^2 =$$

$$= \begin{pmatrix} 2 & -3 \\ \varepsilon & 4 \end{pmatrix} \otimes \begin{pmatrix} 4 & 1 \\ \varepsilon & 8 \end{pmatrix} = \begin{pmatrix} 6 & 5 \\ \varepsilon & 12 \end{pmatrix}$$
Example:

Let us consider the model system with 4 machines introduced in previous examples. Let $A$ be the transition matrix of the given DDS

$$A = \begin{pmatrix}
3 & 4 & \varepsilon & 8 \\
2 & 5 & 2 & \varepsilon \\
4 & 6 & 3 & 4 \\
\varepsilon & 3 & 0 & 6
\end{pmatrix}$$

Suppose that the starting-times for each machine is zero. A particular project involves five stages of activity. At what time will each machine finish work on the project?
Solution: At first stage the machines do not constrain one another. Hence the diagonal elements of the transition matrix $A$ represents the completion times of the first stages $x(1)$:

$$x(1) = \begin{pmatrix} x_1(1) \\ x_2(1) \\ x_3(1) \\ x_4(1) \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 3 \\ 6 \end{pmatrix}$$

Since $x(r+1) = A \otimes x(r) \implies$

$$x(2) = A \otimes x(1)$$
$$x(3) = A \otimes x(2)$$
$$x(4) = A \otimes x(3)$$
$$x(5) = A \otimes x(4)$$

$$\implies x(5) = A^4 \otimes x(1)$$
Max-plus algebra

Operations with vectors and matrices

Application - production line - Example

\[ A^2 = \begin{pmatrix} 3 & 4 & \varepsilon & 8 \\ 2 & 5 & 2 & \varepsilon \\ 4 & 6 & 3 & 4 \\ \varepsilon & 3 & 0 & 6 \end{pmatrix}^2 = \begin{pmatrix} 6 & 11 & 8 & 14 \\ 7 & 10 & 7 & 10 \\ 8 & 11 & 8 & 12 \\ 5 & 9 & 6 & 12 \end{pmatrix} \]

\[ A^4 = (A^2)^2 = \begin{pmatrix} 6 & 11 & 8 & 14 \\ 7 & 10 & 7 & 10 \\ 8 & 11 & 8 & 12 \\ 5 & 9 & 6 & 12 \end{pmatrix}^2 = \begin{pmatrix} 19 & 23 & 20 & 26 \\ 17 & 20 & 17 & 22 \\ 18 & 21 & 18 & 24 \\ 17 & 21 & 18 & 24 \end{pmatrix} \]

\[ x(5) = A^4 \otimes x(1) = \begin{pmatrix} 19 & 23 & 20 & 26 \\ 17 & 20 & 17 & 22 \\ 18 & 21 & 18 & 24 \\ 17 & 21 & 18 & 24 \end{pmatrix} \otimes \begin{pmatrix} 3 \\ 5 \\ 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 32 \\ 28 \\ 30 \\ 30 \end{pmatrix} \]
Theorem

Let \( A \in \mathbb{R}^*(n, n) \). Let \( p \in \mathbb{N}^+ \).

\[
a_{ii}^{(p)} \geq a_{ii}^p \quad \text{for } i = 1, 2, \ldots, n
\]

Proof: \( a_{ii}^{(2)} = a_{i1} \otimes a_{1i} \oplus \cdots \oplus a_{ii} \otimes a_{ii} \oplus \cdots \oplus a_{in} \otimes a_{ni} \geq a_{ii}^2 \)

Remark: The diagonal elements of the powers of the transition matrix have crucial relevance to the question of the maximum speed at which the system can run.
Idempotent law and binomial theorem for matrices

**Theorem**

Let $A \in \mathbb{R}^*(m, n)$. The following holds

\[ A \oplus A = A \quad \text{Idempotent law } \oplus \text{ for matrices} \]

Idempotent law of matrix addition simplifies the binomial theorem for matrices.

**Theorem**

Let $A \in \mathbb{R}^*(n, n)$, $B \in \mathbb{R}^*(n, n)$. Let $p \in \mathbb{N}^+$. The following holds

\[ (A \oplus B)^p = A^p \oplus A^{p-1} \otimes B \oplus \cdots \oplus A \otimes B^{p-1} \oplus B^p. \]

P: by mathematical induction using Idempotent law.
Application of binomial theorem for matrices

Application:

Let $A \in \mathbb{R}^*(n, n)$. Let $E \in \mathbb{R}^*(n, n)$ be the identity matrix. Let us denote by $\Gamma_p$ the sum of identity matrix and $p \in \mathbb{N}$ consecutive powers of the matrix $A$

$$\Gamma_p = E \oplus A \oplus A^2 \oplus \cdots \oplus A^p$$

Using binomial theorem for matrices gives

$$\Gamma_p = (E \oplus A)^p$$

The matrix $\Gamma_p$ will play an important role later on.
**Definition**

Let DDS be given by an initial vector $x \in \mathbb{R}^*(n)$ and a transition matrix $A \in \mathbb{R}^*(n, n)$. The sequence of states $x$, $A \otimes x$, $A^2 \otimes x$, ..., $A^p \otimes x$ is the so-called $(p + 1)$-stage forward orbit based on vector $x(1)$.

The 5-stage forward orbit for the model system from the previous example:

$$
\begin{align*}
  x(1) & = \begin{pmatrix} 3 \\ 5 \\ 3 \\ 6 \end{pmatrix}, \\
  x(2) & = \begin{pmatrix} 14 \\ 10 \\ 11 \\ 12 \end{pmatrix}, \\
  x(3) & = \begin{pmatrix} 20 \\ 16 \\ 18 \\ 18 \end{pmatrix}, \\
  x(4) & = \begin{pmatrix} 26 \\ 22 \\ 24 \\ 24 \end{pmatrix}, \\
  x(5) & = \begin{pmatrix} 32 \\ 28 \\ 30 \\ 30 \end{pmatrix}
\end{align*}
$$
Theorem

$(p + 1)$ - stage orbit of DDS with $n$ machines can be computed with computational complexity $O(pn^2)$.

P:

$x_i(j)$  $n$ operations
$x(j)$  $n^2$ operations
$x(2), x(3), \ldots, x(p + 1)$  $p \times n^2$ operations
Example:

Let us compute effectively for given matrix $A$

- $A^{16}$
- $A^{15}$

Solution:

- $A^{16} = (((A^2)^2)^2)^2 = A^{24}$
- $A^{15} = A^{1+2+4+8} = A \otimes A^2 \otimes A^4 \otimes A^8$
The power $A^p$ for matrix $A \in \mathbb{R}^*(n, n)$ can be computed with computational complexity $O(n^3 \ln p)$.

P:

1. $p = 2^k$

$A^2$, $A^4$, ..., $A^{2^k}$... $k$ - times matrix powering

Matrix power $n^3$ operations

$k$ matrix power $k \times n^3$ operation

$p = 2^k \Rightarrow k = \ln p \Rightarrow kn^3 = n^3 \ln p$
$2\; \; p \neq 2^k$

$A^2, A^4, ..., A^{2^{k+1}-1} \ldots \text{worst case}$

**ALGORITHM:**

\[
\begin{align*}
A & \quad B = A \\
A^2 & \quad B = B \otimes A^2 \\
A^4 & \quad B = B \otimes A^4 \\
\vdots & \\
A^{2^k} & \quad B = B \otimes A^{2^k} \\
\hline
k \times \text{matrix powering} & \quad k \times \text{times matrix powering}
\end{align*}
\]

$\Rightarrow kn^3 + kn^3 = 2kn^3 \quad \Rightarrow \quad n^3 \ln p$
Definition

Let \( m, n \in \mathbb{N} \). The set of all \( m \times n \) matrices in which no row or column contains \( \varepsilon \) exclusively will be denoted by \( F(m, n) \subseteq \mathbb{R}^*(m, n) \).

Remark: \( F(m, 1) \) represents the set of all finite \( m \)-rowed vectors.

Theorem

Let \( A \in F(m, r), \ B \in F(r, n) \). Then \( A \otimes B = C \in F(m, n) \).

\[
P: \quad \forall j \ \exists k \ \exists i; \quad b_{kj} > \varepsilon \ \land \ a_{ik} > \varepsilon \implies c_{ij} = \sum_l a_{il} \otimes b_{lj} \geq a_{ik} \otimes b_{kj} > \varepsilon
\]
Increasing DDS

**Definition**

DDS with forward orbit $x(1), x(2), ...$ will be called **increasing**, if for $i = 1, 2, \ldots, n$ is

$$x_i(r + 1) \geq x_i(r) \quad \text{for} \quad r = 1, 2, \ldots.$$

**Remark:** Systems of the kind we have discussed have the increasing property regardless of the initial vector.

**Theorem**

Let $A \in \mathbb{R}^*(n, n)$ with $a_{ii} \geq 0$, pre $i = 1, 2, \ldots, n$. Then the following holds

1. $A \in F(n, n)$,
2. DDS with transition matrix $A$ is increasing.
Increasing DDS and diagonal elements of transition matrix

P:

1. definition $F(n, n)$

$$x_i(r + 1) = \sum_l a_{il} \otimes x_l(r) \geq a_{ii} \otimes x_i(r) \geq x_i(r)$$

Corollary

Let $A \in \mathbb{R}^*(n, n)$ with $a_{ii} \geq 0$, for $i = 1, 2, \ldots, n$. Then for $r \geq 1$ holds

$$x_i(r + 1) \geq a_{ii}^r \otimes x_i(1).$$

P:

$$x_i(r + 1) \geq a_{ii} \otimes x_i(r) \geq a_{ii} \otimes (a_{ii} \otimes x_i(r - 1)) = a_{ii}^2 \otimes x_i(r - 1) \geq a_{ii}^3 \otimes x_i(r - 2) \geq \cdots \geq a_{ii}^r \otimes x_i(1)$$
Illustration - Example

Example:

Let us consider the model system with 4 machines introduced in previous examples. Let $A$ be the transition matrix of the given DDS

$$
A = \begin{pmatrix}
3 & 4 & \varepsilon & 8 \\
2 & 5 & 2 & \varepsilon \\
4 & 6 & 3 & 4 \\
\varepsilon & 3 & 0 & 6
\end{pmatrix}
$$

and the 5-stage forward orbit with initial event-times zero:

\[
\begin{align*}
x(1) &= \begin{pmatrix} 3 \\ 5 \\ 3 \\ 6 \end{pmatrix}, \\
x(2) &= \begin{pmatrix} 14 \\ 10 \\ 11 \\ 12 \end{pmatrix}, \\
x(3) &= \begin{pmatrix} 20 \\ 16 \\ 18 \\ 18 \end{pmatrix}, \\
x(4) &= \begin{pmatrix} 26 \\ 22 \\ 24 \\ 24 \end{pmatrix}, \\
x(5) &= \begin{pmatrix} 32 \\ 28 \\ 30 \\ 30 \end{pmatrix}
\end{align*}
\]
Thank you for your attention.