GEOMETRIC AND PHYSICAL INTERPRETATION
OF FRACTIONAL INTEGRATION AND
FRACTIONAL DIFFERENTIATION *

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Dedicated to Professor Francesco Mainardi,
on the occasion of his 60-th birthday

Abstract

A solution to the more than 300-years old problem of geometric and physical interpretation of fractional integration and differentiation (i.e., integration and differentiation of an arbitrary real order) is suggested for the Riemann-Liouville fractional integration and differentiation, the Caputo fractional differentiation, the Riesz potential, and the Feller potential. It is also generalized for giving a new geometric and physical interpretation of more general convolution integrals of the Volterra type. Besides this, a new physical interpretation is suggested for the Stieltjes integral.

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1. Introduction

It is generally known that integer-order derivatives and integrals have clear physical and geometric interpretations, which significantly simplify their use for solving applied problems in various fields of science.

However, in case of fractional-order integration and differentiation, which represent a rapidly growing field both in theory and in applications to real-world problems, it is not so. Since the appearance of the idea of differentiation and integration of arbitrary (not necessary integer) order there was not any acceptable geometric and physical interpretation of these operations for more than 300 years. The lack of these interpretations has been acknowledged at the first international conference on the fractional calculus in New Haven (USA) in 1974 by including it in the list of open problems [21]. The question was unanswered, and therefore repeated at the subsequent conferences at the University of Strathclyde (UK) in 1984 [15] and at the Nihon University (Tokyo, Japan) in 1989 [19]. The round-table discussion [13, 10, 14] at the conference on transform methods and special functions in Varna (1996) showed that the problem was still unsolved, and since that time the situation, in fact, still did not change.

Fractional integration and fractional differentiation are generalizations of notions of integer-order integration and differentiation, and include n-th derivatives and n-fold integrals (n denotes an integer number) as particular cases. Because of this, it would be ideal to have such physical and geometric interpretations of fractional-order operators, which will provide also a link to known classical interpretations of integer-order differentiation and integration.

Since the need for the aforementioned geometric and physical interpretations is generally recognised, several authors attempted to provide them. Probably due mostly to linguistical reasons, much effort have been devoted to trying to relate *fractional* integrals and derivatives, on one side, and *fractal geometry*, on the other [18, 27, 9, 16, and others]. However, it has been clearly shown by R. Rutman [22, 23] that this approach is inconsistent.

Besides those "fractal-oriented" attempts, some considerations regarding interpretation of fractional integration and fractional differentiation were presented in [16]. However, those considerations are, in fact, only a small collection of selected examples of applications of fractional calculus, in which hereditary effects and self-similarity are typical for the objects modelled with the help of fractional calculus. Although each particular problem, to which fractional derivatives or/and fractional integrals have been applied, can be

considered as a certain illustration of their meaning, the paper [16] cannot be considered as an explicit answer to the posed question.

A different approach to geometric interpretation of fractional integration and fractional differentiation, based on the idea of the contact of α-th order, has been suggested by F. Ben Adda [1, 2]. However, it is difficult to speak about an acceptable *geometric interpretation* if one cannot see any picture there.

Obviously, there is still a lack of geometric and physical interpretation of fractional integration and differentiation, which is comparable with the simple interpretations of their integer-order counterparts.

In this paper we present a new approach to solution of this challenging old problem.

We start with introducing a simple and really geometric interpretation of several types of fractional-order integration: the left-sided and the right-sided Riemann–Liouville fractional integration, the Riesz potential, and the Feller potential.

Based on this, a physical interpretation of the Riemann–Liouville fractional integration is proposed in terms of inhomogeneous and changing (non-static, dynamic) time scale. Moreover, on this way we give a new physical interpretation of the Stieltjes integral. We also try to persuade the readers that the suggested physical interpretation of fractional integration is in line with the current views on space–time in physics. We also suggest physical interpretation for the Riemann-Liouville fractional differentiation and for the Caputo fractional differentiation. Finally, we show that the suggested approach to geometric interpretation of fractional integration can be used for providing a new geometric and physical interpretation for convolution integrals of the Volterra type.

2. Geometric interpretation of fractional integration: Shadows on the walls

In this section we first give a geometric interpretation of left-sided and right-sided Riemann–Liouville fractional integrals, and then consider the Riesz potential.

2.1. Left-sided Riemann–Liouville fractional integral

Let us consider the left-sided Riemann–Liouville fractional integral [20, 24] of order α,

$$_0I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int\limits_0^t f(\tau)(t-\tau)^{\alpha-1}\,d\tau, \quad t \geq 0, \tag{1}$$

and write it in the form

$$_0I_t^\alpha f(t) = \int\limits_0^t f(\tau)\,dg_t(\tau), \tag{2}$$

$$g_t(\tau) = \frac{1}{\Gamma(\alpha+1)}\Big\{t^\alpha - (t-\tau)^\alpha\Big\}. \tag{3}$$

The function $g_t(\tau)$ has an interesting scaling property. Indeed, if we take $t_1 = kt$ and $\tau_1 = k\tau$ $(k > 0)$, then

$$g_{t_1}(\tau_1) = g_{kt}(k\tau) = k^\alpha g_t(\tau). \tag{4}$$

Now let us consider the integral (2) for a fixed t. Then it becomes simply a Stieltjes integral, and we can utilize G. L. Bullock's idea [3].

Let us take the axes τ, g, and f. In the plane (τ, g) we plot the function $g_t(\tau)$ for $0 \leq \tau \leq t$. Along the obtained curve we "build a fence" of the varying height $f(\tau)$, so the top edge of the "fence" is a three-dimensional line $(\tau, g_t(\tau), f(\tau))$, $0 \leq \tau \leq t$.

This "fence" can be projected onto two surfaces (see Fig. 1):

- the area of the projection of this "fence" onto the plane (τ, f) corresponds to the value of the integral

$$_0I_t^1(t) = \int\limits_0^t f(\tau)\,d\tau; \quad t \geq 0, \tag{5}$$

- the area of the projection of the same "fence" onto the plane (g, f) corresponds to the value of the integral (2), or, what is the same, to the value of the fractional integral (1).

In other words, our "fence" throws two shadows on two walls. The first of them, that on the wall (τ, f), is the well-known "area under the curve $f(\tau)$", which is a standard geometric interpretation of the integral (5). The "shadow" on the wall (g, f) is a geometric interpretation of the fractional integral (1) for a fixed t.
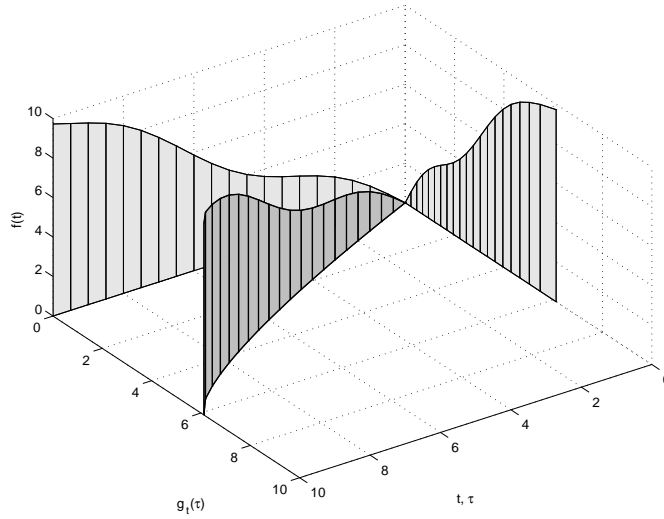
Figure 1: The "fence" and its shadows: ${}_0I_t^1 f(t)$ and ${}_0I_t^\alpha f(t)$, for $\alpha = 0.75$, $f(t) = t + 0.5\sin(t)$, $0 \leq t \leq 10$.
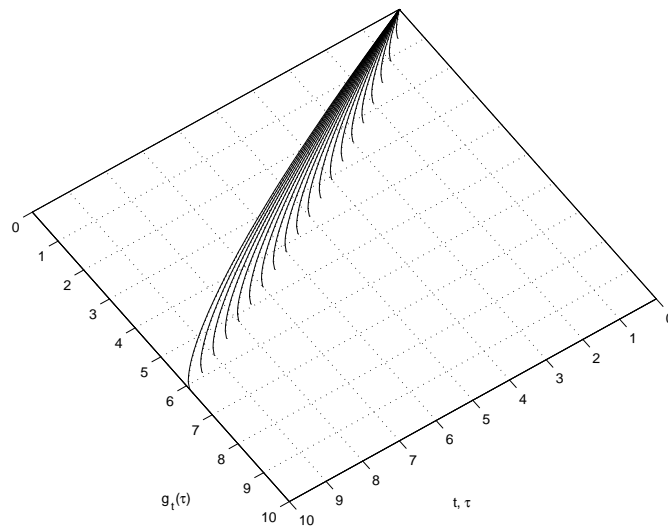

Figure 2: The process of change of the fence basis shape for ${}_0I_t^\alpha f(t)$, $\alpha = 0.75$, $0 \leq t \leq 10$.
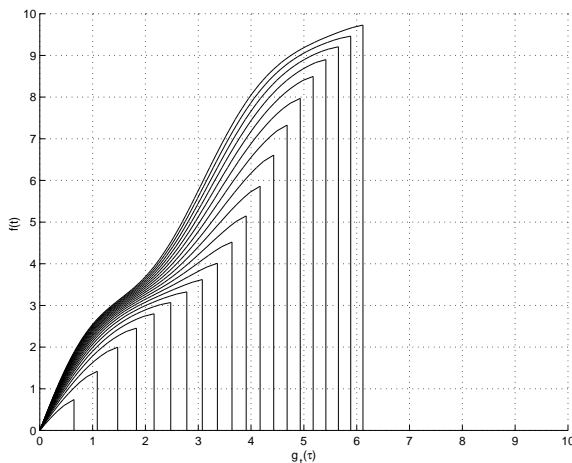
Figure 3: Snapshots of the changing "shadow" of changing "fence" for $_0I_t^\alpha f(t)$, $\alpha = 0.75$, $f(t) = t + 0.5\sin(t)$, $0 \le t \le 10$, with the time interval $\Delta t = 0.5$ between the snapshops.

Obviously, if $\alpha = 1$, then $g_t(\tau) = \tau$, and both "shadows" are equal. This shows that classical definite integration is a particular case of the left-sided Riemann–Liouville fractional integration even from the geometric point of view.

What happens when t is changing (namely growing)? As t changes, the "fence" changes simultaneously. Its length and, in a certain sense, its shape changes. For illustration, see Fig. 2. If we follow the change of the "shadow" on the wall (g, f), which is changing simultaneously with the "fence" (see Fig. 3), then we have a dynamical geometric interpretation of the fractional integral (1) as a function of the variable t.

2.2. Right-sided Riemann–Liouville fractional integral

Let us consider the right-sided Riemann–Liouville fractional integral [20, 24],

$$_tI_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int\limits_t^b f(\tau)(\tau - t)^{\alpha-1} \, d\tau, \quad t \le b, \tag{6}$$

and write it in the form

$$_tI_0^\alpha f(t) = \int_t^b f(\tau)\,dh_t(\tau), \tag{7}$$

$$h_t(\tau) = \frac{1}{\Gamma(\alpha+1)}\left\{t^\alpha + (\tau-t)^\alpha\right\}. \tag{8}$$

Then we can provide a geometric interpretation similar to the geometric interpretation of the left-sided Riemann–Liouville fractional integral. However, in this case there is not any fixed point in the "fence" base – the end, corresponding to $\tau = b$, moves along the line $\tau = b$ in the plane (τ, g) when the "fence" changes its shape. This movement can be observed in Fig. 4. (In the case of the left-sided integral, the left end, corresponding to $\tau = 0$, is fixed and does not move.)

All other parts of the geometric interpretation remain the same: the "fence" changes its shape as t changes from 0 to b, and the changing shadows of this "fence" on the walls (g, f) and (τ, f) represent correspondingly the right-sided Riemann–Liouville fractional integral (6) and the classical integral with the moving lower limit:

$$_tI_b^1(t) = \int_t^b f(\tau)\,d\tau; \tag{9}$$

Obviously, if $\alpha = 1$, then $h_t(\tau) = \tau$, and both "shadows" are equal. Therefore, we see that not only the left-sided, but also the right-sided Riemann-Liouville fractional integration includes the classical definite integration as a particular case even from the geometrical point of view.

2.3. Riesz potential

The Riesz potential [20, 24]

$$_0R_b^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^b f(\tau)|\tau-t|^{\alpha-1}\,d\tau, \quad 0 \le t \le b, \tag{10}$$

is the sum of the left-sided and the right-sided Riemann–Liouville fractional integrals:

$$_0R_b^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t f(\tau)(t-\tau)^{\alpha-1}\,d\tau + \frac{1}{\Gamma(\alpha)}\int_t^b f(\tau)(\tau-t)^{\alpha-1}\,d\tau. \tag{11}$$
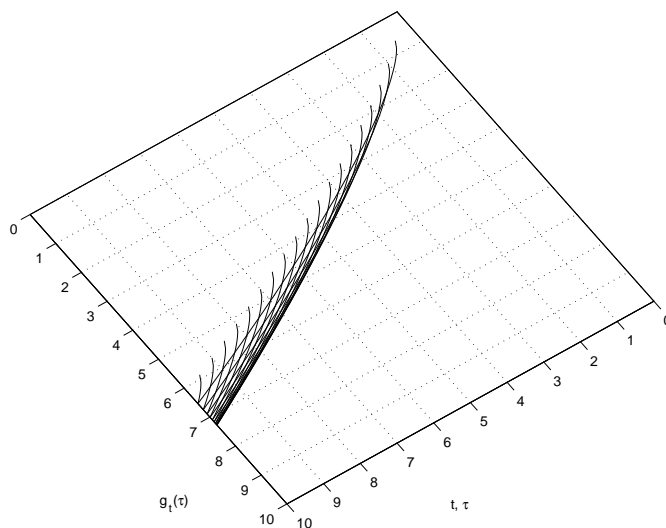
Figure 4: The process of change of the fence basis shape
for $_tI_{10}^\alpha f(t)$, $\alpha = 0.75$, $0 \le t \le 10$.

The Riesz potential (10) can be written in the form

$$_0R_b^\alpha f(t) = \int\limits_0^b f(\tau)\, dr_t(\tau), \tag{12}$$

$$r_t(\tau) = \frac{1}{\Gamma(\alpha+1)}\Big\{t^\alpha + \text{sign}(\tau - t)\,|\tau - t|^\alpha\Big\}. \tag{13}$$

The shape of the "fence", corresponding to the Riesz potential, is described by the function $r_t(\tau)$. In this case the "fence" consists of the two parts: one of them (for $0 < \tau < t$) is the same as in the case of the left-sided Riemann–Liouville fractional integral, and the second (for $t < \tau < b$) is the same as for the right-sided Riemann–Liouville integral, as shown in Fig. 5. Both parts are joined smothly at the inflection point $\tau = t$.

The shape of the "fence", corresponding to the Riesz potential, is shown in some of its intermediate position by the bold line in Fig. 5. Obviously, Fig. 5 can be obtained by laying Fig. 4 over Fig. 2, which is a geometric interpretation of the relationship (11).

The shadow of this "fence" on the wall (g, f) represents the Riesz potential (10), while the shadow on the wall (τ, f) corresponds to the classical
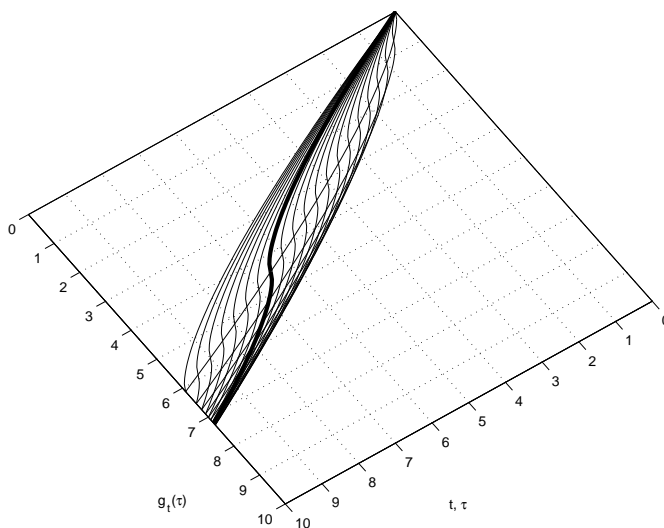
Figure 5: The process of change of the fence basis shape
for the Riesz potential $_0R_{10}^\alpha f(t)$, $\alpha = 0.75$, $0 \le t \le 10$.

integral

$$I(t) = \int\limits_0^b f(\tau)\,d\tau. \tag{14}$$

For $\alpha = 1$, $r_t(\tau) = \tau$, and both "shadows" are equal. This shows that the classical definite integral (14) is a particular case of the Riesz fractional potential (10) even from the geometric point of view. We have already seen this inclusion in the case of the left-sided and the right-sided Riemann–Liouville fractional integration. This demonstrates the strength of the suggested geometric interpretation of these three types of generalization of the notion of integration.

2.4. Feller potential

The Feller potential operator $\Phi^\alpha f(t)$ is, similarly to the Riesz potential, also a linear combination of the left- and right-sided Riemann–Liouville fractional integrals, but with general constant coefficients c, d [24, Chap. 3]:

$$\Phi^\alpha f(t) = c\,_aI_t^\alpha f(t) + d\,_tI_b^\alpha f(t). \tag{15}$$

The geometric interpretation of the Feller potential can be easily obtained by properly scaling and then superimposing Fig. 4 and Fig. 2. The "fence" obtained in this way is, in general, discontinuous at $\tau = t$. Its shadow on the wall (τ, f) is equal to the classical definite integral (14). The shadow on the wall (g, f) consists, in general, of the two areas, which may overlap depending on the values of the coefficients c and d.

3. Two kinds of time – I

The geometric interpretation of fractional integration, given in the previous sections, is substantially based on adding the third dimension (for $g_t(\tau)$) to the classical pair τ, $f(\tau)$. If we consider τ as time, then $g(\tau)$ can be interpreted just as a "transformed" time scale. What could be the meaning of having – and using – two time axes? To answer this question, let us recall some facts of the history of the development of the notion of time.

That were contributions of Barrows and Newton to the development of mathematics and physics in the XVII century which led to the appearance of the "mathematical time", which is postulated to "flow equably" and which is usually depicted as a semi-infinite straight line [26].

Newton himself postulated [17]:

> "Absolute, true and mathematical time of itself, and from its own nature, flows equably without relation to anything external."

Such a postulate was absolutely necessary for developing Newton's differential calculus and applying it to problems of mechanics [26]:

> "The outstanding mathematical achievement associated with the geometrization of time was, of course, the invention of the calculus of fluxions by Newton."
>
> "Mathematically, Newton seems to have found support for his belief in absolute time by the need, in principle, for an ideal rate-measurer."

The invention of differential and integral calculus and today's use of them is the strongest reason for continuing using homogeneous equably flowing time.

Figure 6: Homogeneous time axes.

Figure 7: Time slowing down.

Time is often depicted using the time axis, and the geometrically equal intervals of the time axis are considered as corresponding to equal time intervals (Fig. 6).

This assumption, however, cannot be neither proved nor rejected by experiment. Two lengths of geometric intervals can be measured and compared, since they are available for measurement simultaneously, at the same time (or, more precisely, at the same time and at the same place). Two time intervals can never be compared, because they are available to us for measurement (or for observation) only sequentially.

Indeed, how do we measure time intervals? Only by observing some processes, which we consider as regularly repeated. G. Clemence wrote [7]:

> "The measurement of time is essentially a process of counting. Any recurring phenomenon whatever, the occurences of which can be counted, is in fact a measure of time."

Clocks, including atomic clocks, repeat their "ticks", and we simply count those ticks, calling them hours, minutes, seconds, milliseconds, etc. But we are not able to verify if the *absolute* time which elapsed between, say, the fifth and the sixth tick (the sixth "second") is exactly the same as the time, which elapsed between the sixth and the seventh tick (the seventh "second"). This possible inhomogeneity of the time scale is illustrated in Fig. 7.

The fact that time measurement as a process of counting of repeating discrete events does not really exclude inhomogeneity of time, has been nicely mentioned by L. Carroll in *Alice's Adventures in Wonderland* [6, Chap. 7]:

> "...I know I have to beat time when I learn music."

Table 1: Recording speed using slowing-down clocks

Person N individual "seconds"	Recorded values of velocity [m/s]	Observer O absolute (cosmic) "seconds"
0	10	0
1	11	1
2	12	3
3	13	7
4	12	15
5	11	31
6	10	63
7	9	127

"Ah! That accounts for it," said the Hatter. "He [Time] won't stand beating. Now, if you only kept on good terms with him, he'd do almost anything you liked to do with the clock..."

Figures 6 and 7 show those "clock ticks", which we can register, only symbolically. One can interpret them as if there exists some *absolute*, or *cosmic*, inhomogeneous time axis, to which we can compare *individual* homogeneous time, represented by some "clock ticks". Our picture of the individual homogeneous time has the form shown in Fig. 6. The cosmic time may be not necessarily flowing equably, like that shown in Fig. 7.

To illustrate the idea, let us consider the following situation. Suppose person N has two devices: one is a speedometer, and another one is the clock, which is slowing down, so the interval between the two subsequent ticks is double comparing to the interval between the previous ticks (see Fig. 7). Person N reads the velocity values indicated by the speedometer at each encountered "second", without knowing that the clock is, in fact, slowing down.

Using these two series of data, namely the recorded sequence of values of speed, and the sequence of the counted "seconds", person N can estimate the distance which he has passed.

For simplicity, let us suppose that the first "second" of the time shown by the clocks is equal to the absolute time "second". The results of observations in this hypothetical experiment are given in Table 1.

Person N will compute the distance he has passed as

$$S_N = 10 \cdot 1 + 11 \cdot 1 + 12 \cdot 1 + 13 \cdot 1 + 12 \cdot 1 + 11 \cdot 1 + 10 \cdot 1 = 79.$$

However, if there would be an independent observer O, knowing about the slowing-down clock, then such an observer would obtain a notably different result for the distance passed by person N:

$$S_O = 10 \cdot 1 + 11 \cdot 2 + 12 \cdot 4 + 13 \cdot 8 + 12 \cdot 16 + 11 \cdot 32 + 10 \cdot 64 = 1368.$$

Below we use this idea for giving a new mechanical interpretation of the Stieltjes integral.

4. Physical interpretation of the Stieltjes integral

Imagine a car equipped with two devices for measurements: the speedometer recording the velocity $v(\tau)$, and the clock which should show the time τ. The clock, however, shows the time incorrectly; let us suppose that the relationship between the wrong time τ, which is shown by the clock and which the driver considers as the correct time, on one hand, and the true time T, on the other, is described by the function $T = g(\tau)$. This means that where the driver "measures" the time interval $d\tau$, the real time interval is given by $dT = dg(\tau)$.

The driver A, who do not know about wrong operation of the clock, will compute the passed distance as the classical integral:

$$S_A(t) = \int\limits_0^t v(\tau)\, d\tau\,. \tag{16}$$

However, the observer O knowing about the wrong clock and having the function $g(\tau)$, which restores the correct values of time from the driver's wrong time τ, will compute the really passed distance as

$$S_O(t) = \int\limits_0^t v(\tau)\, dg(\tau). \tag{17}$$

This example shows that the Stieltjes integral (17) can be interpreted as the real distance passed by a moving object, for which we have recorded correct values of speed and incorrect values of time; the relationship between the wrongly recorded time τ and the correct time T is given by a known function $T = g(\tau)$.

5. Physical interpretation of fractional integration:
Shadows of the past

Now let us consider the left-sided Riemann–Liouville fractional integral

$$S_O(t) = \int\limits_0^t v(\tau)\,dg_t(\tau) = {}_0I_t^{\alpha}v(t), \qquad (18)$$

where $g_t(\tau)$ is given by (3).

The fractional integral $S_O(t)$ of the function $v(\tau)$ can be interpreted as the real distance passed by a moving object, for which we have recorded the local values of its speed $v(\tau)$ (individual speed) and the local values of its time τ (individual time); the relationship between the locally recorded time τ (which is considered as flowing equably) and the cosmic time (which flows non-equably) is given by a known function $g_t(\tau)$.

The function $g_t(\tau)$ describes the inhomogeneous time scale, which depends not only on τ, but also on the parameter t representing the last measured value of the individual time of the moving object. When t changes, the entire preceding cosmic time interval changes as well. This is in agreement with the current views in physics. Indeed, B. N. Ivanov [12, p. 33] mentioned that time intervals depend on gravitational fields. Similarly, S. Hawking [11, p. 32–33] wrote that:

> "... time should appear to run slower near a massive body like the earth." [...]
>
> "... there is no unique absolute time, but instead each individual has his own personal measure of time that depends on where he is and how he is moving."

When a moving body changes its position in space–time, the gravitational field in the entire space–time also changes due to this movement. As a consequence, the cosmic time interval, which corresponds to the history of the movement of the moving object, changes. This affects the calculation (using formula (18)) of the real distance $S_O(t)$ passed by such a moving object.

In other words, the left-sided Riemann–Liouville fractional integral of the individual speed $v(\tau)$ of a moving object, for which the relationship between its individual time τ and the cosmic time T at each individual time instance t is given by the known function $T = g_t(\tau)$ described by the equation (3), represents the real distance $S_O(t)$ passed by that object.

6. Physical interpretation of the Riemann-Liouville fractional derivative

On the other hand, we can use the properties of fractional differentiation and integration [20, 24] and express $v(t)$ from the equation (18) as a left-sided Riemann–Liouville fractional derivative of $S_O(t)$:

$$v(t) = \ _0D_t^\alpha S_O(t) \tag{19}$$

where $_0D_t^\alpha$ denotes the Riemann–Liouville fractional derivative [20, 24], which is for $0 < \alpha < 1$ defined by

$$_0D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int\limits_0^t \frac{f(\tau)\,d\tau}{(t-\tau)^\alpha}. \tag{20}$$

This shows that the left-sided Riemann–Liouville fractional derivative of the real distance $S_O(t)$ passed by a moving object, for which the relationship between its individual time τ and the cosmic time T at each individual time instance t is given by the known function $T = g_t(\tau)$ described by equation (3), is equal to the individual speed $v(\tau)$ of that object.

On the other hand, we can differentiate the relationship (18) with respect to the cosmic time variable t, which gives the relationship between the velocity $v_O(t) = S_O'(t)$ of the movement from the viewpoint of the independent observer O and the individual velocity $v(t)$:

$$v_O(t) = \frac{d}{dt}\ _0I_t^\alpha v(t) = \ _0D_t^{1-\alpha}v(t), \tag{21}$$

Therefore, the $(1 - \alpha)$-th–order Riemann–Liouville derivative of the individual velocity $v(t)$ is equal to the velocity $v_O(t)$ from the viewpoint of the independent observer, if the individual time τ and the cosmic time T are related by the function $T = g_t(\tau)$ described by equation (3). For $\alpha = 1$, when there is no dynamic deformation of the time scale, both velocities coincide: $v_O(t) = v(t)$.

7. Physical interpretation of the Caputo fractional derivative

Applying fractional integration of order $\beta = 1 - \alpha$ to both parts of the relationship (21) gives:

$$v(t) = \ _0I_t^{1-\alpha}v_O(t) = \ _0I_t^{1-\alpha}S_O'(t) = \ _0^C D_t^\alpha S_O(t), \tag{22}$$

where $_{0}^{C}D_{t}^{\alpha}$ denotes the Caputo fractional derivative [4, 5, 20], which is for $0 < \alpha < 1$ defined by

$$_{0}^{C}D_{t}^{\alpha}f(t) = \frac{1}{\Gamma(1-\alpha)} \int\limits_{0}^{t} \frac{f'(\tau)\,d\tau}{(t-\tau)^{\alpha}}. \tag{23}$$

The relationship (22) is similar to (19). Therefore, the Caputo fractional derivative has the same physical interpretation as the Riemann–Liouville fractional derivative (see Section 6). This coincidence becomes more obvious, if we recall [20] that if $f(0) = 0$, then the Riemann–Liouville derivative and the Caputo derivative of order α $(0 < \alpha < 1)$, coincide: $_{0}^{C}D_{t}^{\alpha}f(t) = {}_{0}D_{t}^{\alpha}f(t)$.

8. Two kinds of time – II

The suggested physical interpretation of fractional integration and fractional differentiation is based on using two kinds of time: the cosmic time and the individual time.

As mentioned above, due to the history of the development of mathematics and physics, we are taught to think about the time, in fact, geometrically. The real roots of this go even far back to Euclid [26]:

> "Euclid considered space as the primary concept of science and relegated time to poor second."

The entire integral and differential calculus is based on using mathematical (homogeneous, equably flowing) time. There is no chance to change this state, and there is nothing to suggest instead of the classical calculus. Moreover, there is probably even no need for this. We can just realize that the classical calculus provides tools for describing the dynamic properties of the cosmic time, which – according to physicists – is inhomogeneous (flowing non-equably). Indeed [11, p. 33–34],

> "The old idea of an essentially unchanging universe that could have existed, and could continue to exist, forever was replaced by the notion of a dynamic, expanding universe that seemed to have begun a finite time ago..."

Clearly, the expansion of the universe implies that neither spatial scale nor time scale remains homogeneous; they both are dynamic. For describing the inhomogeneous time, the ideal homogeneous time scale can be used.

This approach is not new; it has already been used in the theory of relativity for describing shortening of time intervals. This means that in fact two time scales are considered simultaneously: the ideal, equably flowing homogeneous time, and the cosmic (inhomogeneous) time. The change of scale of the cosmic time is described using the homogeneous time scale as a reference scale. In other words, the homogeneous time scale is just an ideal notion, which is necessary for developing mathematical models describing inhomogeneous cosmic time and its change. In this respect we can, without discussing other views on this subject, recall the remark made by A. Daigneault and A. Sangalli in their essay [8] about I. E. Segal and his two-time cosmology ("chronometric cosmology", or CC) [25] – note that "*perhaps!*":

> "According to CC, Einstein's model is the correct one to understand the universe as a whole (i.e., global space–time), except that there are two kinds of time: a cosmic or Einstein's time t, and a local or Minkowski's time x_0, which is (perhaps!) the time measured by existing techniques. [...] Simply put, Einstein's cosmic time is the "real" one, whereas Minkowski's time is only an approximation of t."

So, the ideal model of equably flowing homogeneous time can be considered as a rough approximation of the cosmic time.

9. Geometric and physical interpretation of the Volterra convolution integral

It should be mentioned that we can also provide a geometric and physical interpretation for more general integrals.

The Riemann–Liouville fractional integral is a particular case of convolution integrals of the Volterra type:

$$K * f(t) = \int_0^t f(\tau) k(t - \tau) \, d\tau \tag{24}$$

Assuming that $k(t) = K'(t)$, we can write this integral in the form

$$K * f(t) = \int_0^t f(\tau) \, dq_t(\tau), \tag{25}$$

$$q_t(\tau) = K(t) - K(t - \tau). \tag{26}$$

The geometric and physical interpretation of the Volterra convolution integral is then similar to the suggested interpretations for fractional integrals. The function $q_t(\tau)$ determines the changing shape of the "live fence" (in the case of the geometric interpretation, see Figs. 1 and 2) and the relationship between the individual time and the cosmic time of a moving object (in the case of the physical interpretation).

Acknowledgment

References

[1] F. B e n A d d a, Geometric interpretation of the fractional derivative. *Journal of Fractional Calculus* **11** (May 1997), 21-52.

[2] F. B e n A d d a, Interprétation géometrique de la différentiabilité et du gradient d'ordre réel. *C. R. Acad. Sci. Paris* **326**, Série I (1998), 931-934.

[3] G. L. B u l l o c k, A geometric interpretation of the Riemann–Stieltjes integral. *American Mathematical Monthly* **95**, No 5 (May 1988), 448-455.

[4] M. C a p u t o, Linear model of dissipation whose Q is almost frequency independent – II. *Geophys. J. R. Astr. Soc.* **13** (1967), 529-539.

[5] M. C a p u t o, *Elasticità e Dissipazione.* Zanichelli, Bologna (1969).

[6] L. C a r r o l l, *Alice's Adventures in Wonderland.* Progress, Moscow (1979) (Or, any other edition).

[7] G. C l e m e n c e, Time and its measurement. *The American Scientist* **40** (1952), 260-269.

[8] A. D a i g n e a u l t and A. S a n g a l l i, Einstein's static universe: An idea whose time has come back? *Notices of AMS* **48**, No 1 (2001), 9-16.

[9] F u - Y a o R e n, Z u - G u o Y u, F e n g S u, Fractional integral associated to the self-similar set of the generalised self-similar set and its physical interpretation. *Phys. Lett. A* **219** (1996), 59-68.

[10] R. G o r e n f l o, Afterthoughts on interpretation of fractional derivatives and integrals. In: P. Rusev, I. Dimovski, V. Kiryakova (Eds.), *Transform Methods and Special Functions, Varna'96 (Proc. 3rd Internat. Workshop)*, Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Sofia (1998, ISBN 954-8986-05-1), 589-591.

[11] S. W. H a w k i n g, *A Brief History of Time.* Bantam Press, London (1988).

[12] B. N. I v a n o v, *Printsipy sovremennoi fiziki (Principles of Modern Physics).* Nauka, Moscow (1973) (In Russian).

[13] V. K i r y a k o v a, A long standing conjecture failed? In: P. Rusev, I. Dimovski, V. Kiryakova (Eds.), *Transform Methods and Special Functions, Varna'96 (Proc. 3rd Internat. Workshop)*, Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Sofia (1998, ISBN 954-8986-05-1), 579-588.

[14] F. M a i n a r d i, Considerations on fractional calculus: interpretations and applications. In: P. Rusev, I. Dimovski, V. Kiryakova (Eds.), *Transform Methods and Special Functions, Varna'96 (Proc. 3rd Internat. Workshop)*, Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Sofia (1998, ISBN 954-8986-05-1), 594-597.

[15] A. M c B r i d e, G. R o a c h (Eds), *Fractional Calculus.* Research Notes in Mathematics, vol. **138**, Pitman, Boston–London–Melbourne (1985).

[16] M. M o n s r e f i - T o r b a t i and J. K. H a m m o n d, Physical and geometrical interpretation of fractional operators. *J. Franklin Inst.* **335B**, No 6 (1998), 1077-1086.

[17] I. N e w t o n, *Mathematical Principles* (transl. A. Motte, Ed. F. Cajori), Berkeley (1934).

[18] R. R. N i g m a t u l l i n, A fractional integral and its physical interpretation. *Theoret. and Math. Phys.* **90**, No 3 (1992), 242-251.

[19] K. N i s h i m o t o (Ed.), *Fractional Calculus and Its Applications.* Nihon University, Koriyama (1990).

[20] I. P o d l u b n y, *Fractional Differential Equations.* Academic Press, San Diego (1999).

[21] B. R o s s (Ed.), *Fractional Calculus and Its Applications.* Lecture Notes in Mathematics, vol. **457**, Springer-Verlag, New York (1975).

[22] R. S. R u t m a n, On the paper by R.R. Nigmatullin "A fractional integral and its physical interpretation". *Theoret. and Math. Phys.* **100**, No 3 (1994), 1154-1156.

[23] R. S. R u t m a n, On physical interpretations of fractional integration and differentiation. *Theoret. and Math. Phys.* **105**, No 3 (1995), 1509-1519.

[24] S. G. S a m k o, A. A. K i l b a s, and O. I. M a r i c h e v, *Integraly i proizvodnye drobnogo poryadka i nekotorye ich prilozheniya (Integrals and Derivatives of the Fractional Order and Some of Their Applications).* Nauka i Tekhnika, Minsk (1987) (English transl.: S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives.* Gordon and Breach, Amsterdam (1993)).

[25] I. E. S e g a l, *Mathematical Cosmology and Extragalactic Astronomy.* Academic Press, New York (1976).

[26] G. J. W h i t r o w, *The Natural Philosophy of Time.* Thomas Nelson and Sons Ltd., London and Edinburgh (1961).

[27] Z u - G u o Y u, F u - Y a o R e n, and J i Z h o u, Fractional integral associated to generalized cookie-cutter set and its physical interpretation. *J. Phys.A: Math. Gen.* **30** (1997), 5569-5577.

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