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MATRIX APPROACH TO DISCRETE FRACTIONAL CALCULUS

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Dedicated to Prof. Rudolf Gorenflo,
on the occasion of his 70-th birthday

Abstract

A matrix form representation of discrete analogues of various forms of fractional differentiation and fractional integration is suggested.

The approach, which is described in this paper, unifies the numerical differentiation of integer order and the n-fold integration, using the so-called triangular strip matrices. Applied to numerical solution of differential equations, it also unifies the solution of ordinary integer- and fractional-order differential equations, and of fractional integral equations.

The suggested approach leads to significant simplification of the numerical solution of fractional integral and differential equations.

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1. Introduction

There are several well-known approaches to unification of notions of differentiation and integration, and their extension to non-integer orders [13].

The approach, which is described in this paper, unifies the *numerical* differentiation of integer order and the n-fold integration, using the so-called triangular

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strip matrices [1, 8, 14]. Applied to numerical solution of differential equations, it also unifies solution of ordinary integer- and fractional-order differential equations, and of fractional integral equations.

Triangular strip matrices already appeared in some studies on fractional integral equations [2, 3, 6, 7, 9, 10, 11], but until today their usefulness for approximating fractional derivatives and solving fractional differential equations has not been recognized.

The structure of this paper is the following. First of all, triangular strip matrices, and operations on them, are introduced. Then discrete forms of the integer-order differentiation and of the n-fold integration are considered using triangular strip matrices, and a generalisation for the case of an arbitrary (non-integer) order of differentiation and integration is presented. The advantages of the use of triangular strip matrices for numerical solution of fractional integral and differential equations of some important types are described and illustrated with four examples.

2. Triangular strip matrices

In this paper we deal with matrices of a specific structure, which are called *triangular strip matrices* [14, p. 20], and which have been mentioned in [1, 8]. We will consider lower triangular strip matrices,

$$L_N = \begin{bmatrix} \omega_0 & 0 & 0 & 0 & \cdots & 0 \\ \omega_1 & \omega_0 & 0 & 0 & \cdots & 0 \\ \omega_2 & \omega_1 & \omega_0 & 0 & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \cdots & \cdots \\ \omega_{N-1} & \ddots & \omega_2 & \omega_1 & \omega_0 & 0 \\ \omega_N & \omega_{N-1} & \ddots & \omega_2 & \omega_1 & \omega_0 \end{bmatrix}, \tag{1}$$

and upper triangular strip matrices,

$$U_N = \begin{bmatrix} \omega_0 & \omega_1 & \omega_2 & \ddots & \omega_{N-1} & \omega_N \\ 0 & \omega_0 & \omega_1 & \ddots & \ddots & \omega_{N-1} \\ 0 & 0 & \omega_0 & \ddots & \omega_2 & \ddots \\ 0 & 0 & 0 & \ddots & \omega_1 & \omega_2 \\ \cdots & \cdots & \cdots & \cdots & \omega_0 & \omega_1 \\ 0 & 0 & 0 & \cdots & 0 & \omega_0 \end{bmatrix}. \tag{2}$$

A lower (upper) triangular strip matrix is completely described by its first column (row). Because of this, it may be convenient in the future to use a compact notation of the form

$$L_N = ||\omega_0, \omega_1, \ldots, \omega_N||^T,$$

$$U_N = ||\omega_0, \omega_1, \ldots, \omega_N||,$$

where $|| \cdot ||^T$ denotes matrix transposition. However, in this paper we prefer to use full matrix notation for clarity.

Obviously, if matrices C and D are both lower (upper) triangular strip matrices, then they commute:

$$CD = DC. \tag{3}$$

Denoting

$$\Omega_N = L_N - \omega_0 E, \qquad \Psi_N = U_N - \omega_0 E, \tag{4}$$

where E is the unit matrix, we can write

$$L_N = \omega_0 E + \Omega_N, \qquad U_N = \omega_0 E + \Psi_N. \tag{5}$$

We can also consider $(N+1) \times (N+1)$ matrices E_p^+, $p = 1, \ldots N$, with ones on p-th diagonal above the main diagonal and zeroes elsewhere, and matrices E_p^-, $p = 1, \ldots N$, with ones on p-th diagonal below the main diagonal and zeroes elsewhere. We will also denote $E_0^\pm \equiv E$ the unit matrix.

It can be shown that

$$E_p^\pm E_q^\pm = \begin{cases} E_{p+q}^\pm, & (p+q \le N), \\ O, & (p+q > N), \end{cases} \tag{6}$$

from which follows that for integer k

$$(E_p^\pm)^k = \begin{cases} E_{pk}^\pm, & (pk \le N), \\ O, & (pk > N), \end{cases} \qquad (E_1^\pm)^{N+1} = O. \tag{7}$$

Noting that

$$\Omega_N = \sum_{k=1}^{N} \omega_k E_k^-, \qquad \Psi_N = \sum_{k=1}^{N} \omega_k E_k^+, \tag{8}$$

it can be shown that $(N+1)$-th power of Ω_N and of Ψ_N gives the zero matrix:

$$\Omega_N^{N+1} = O, \qquad \Psi_N^{N+1} = O. \tag{9}$$

Using (9) it is easy to check that the inverse matrices $(L_N)^{-1}$ and $(U_N)^{-1}$ are given by the following explicit expressions [5, p. 62]:

$$(L_N)^{-1} = \omega_0^{-1} E - \omega_0^{-2} \Omega_N + \omega_0^{-3} \Omega_N^2 + \ldots + (-1)^N \omega_0^{-N-1} \Omega_N^N, \tag{10}$$

$$(U_N)^{-1} = \omega_0^{-1} E - \omega_0^{-2} \Psi_N + \omega_0^{-3} \Psi_N^2 + \ldots + (-1)^N \omega_0^{-N-1} \Psi_N^N. \tag{11}$$

There is a link between the matrix polynomials and the triangular strip matrices. Namely, if we introduce the polynomial $\varrho_N(z)$,

$$\varrho_N(z) = \omega_0 + \omega_1 z + \omega_2 z^2 + \ldots + \omega_N z^N, \tag{12}$$

and take into account the relationship (7), then we can write:

$$\varrho_N(E_1^-) = \omega_0 E + \omega_1 E_1^- + \omega_2 (E_1^-)^2 + \ldots + \omega_N (E_1^-)^N = L_N, \tag{13}$$

$$\varrho_N(E_1^+) = \omega_0 E + \omega_1 E_1^+ + \omega_2 (E_1^+)^2 + \ldots + \omega_N (E_1^+)^N = U_N, \tag{14}$$

where L_N and U_N defined by relationships (1) and (2).

If we define the truncation operation, $\mathrm{trunc}_N(\cdot)$, which truncates (in a general case) the power series $\varrho(z)$,

$$\varrho(z) = \sum_{k=0}^{\infty} \omega_k z^k \tag{15}$$

to the polynomial $\varrho_N(z)$,

$$\mathrm{trunc}_N(\varrho(z)) \stackrel{\mathrm{def}}{=} \sum_{k=0}^{N} \omega_k z^k = \varrho_N(z), \tag{16}$$

then we can consider the function $\varrho(z)$ as a generating series for the set of lower (or upper) triangular matrices L_N (or U_N), $N = 1, 2, \ldots$.

We will need the following properties of the truncation operation:

$$\begin{aligned} \mathrm{trunc}_N(\gamma\lambda(z)) &= \gamma\, \mathrm{trunc}_N(\lambda(z)), && (17)\\ \mathrm{trunc}_N(\lambda(z)+\mu(z)) &= \mathrm{trunc}_N(\lambda(z)) + \mathrm{trunc}_N(\mu(z)), && (18)\\ \mathrm{trunc}_N(\lambda(z)\mu(z)) &= \mathrm{trunc}_N\Big(\mathrm{trunc}_N(\lambda(z))\ \mathrm{trunc}_N(\mu(z))\Big). && (19) \end{aligned}$$

3. Operations with triangular strip matrices

Due to the special structure of triangular strip matrices, the operations with them, such as addition, subtraction, multiplication, and inversion, can be expressed in the form of operations with their generating series (15).

Let us consider two $(N+1) \times (N+1)$ lower triangular strip matrices: matrix A_N with elements a_k, $k = 0, 1, \ldots, N$ in its first column, and matrix B_N with elements b_k, $k = 0, 1, \ldots, N$ in its first column. Denoting $\lambda(z)$ and $\mu(z)$ the generating series of A_N and B_N respectively, and using the representation (13), we can write:

$$A_N = \sum_{k=0}^{N} a_k (E_1^-)^k = \lambda_N(E_1^-), \qquad B_N = \sum_{k=0}^{N} b_k (E_1^-)^k = \mu_N(E_1^-), \tag{20}$$

where $\lambda_N(z) = \mathrm{trunc}_N(\lambda(z))$, $\mu_N = \mathrm{trunc}_N(\mu(z))$, and therefore

$$A_N \pm B_N = \sum_{k=0}^{N} (a_k \pm b_k)(E_1^-)^k. \tag{21}$$

In symbolic form, using the generating series $\lambda(z)$ and $\mu(z)$ and the properties of truncation operation (17) and (18), this can be written as

$$\begin{aligned} A_N \pm B_N &\longleftrightarrow \operatorname{trunc}_N(\lambda(z) \pm \mu(z)) = \lambda_N(z) \pm \mu_N(z) = \\ &= \sum_{k=0}^{N} (a_k \pm b_k) z^k \longleftrightarrow \sum_{k=0}^{N} (a_k \pm b_k)(E_1^-)^k. \end{aligned} \tag{22}$$

This means that the coefficient on k-th diagonal of the sum of two lower triangular strip matrices is equal to the sum of k-th coefficients of the generating series of those matrices. Therefore, summation of lower triangular strip matrices is equivalent to summation of their respective generating series with a subsequent truncation.

The multiplication by a constant γ is simple:

$$\gamma A_N \longleftrightarrow \operatorname{trunc}_N(\gamma\lambda(z)) = \gamma\lambda_N(z) = \sum_{k=0}^{N} \gamma a_k z^k \longleftrightarrow \sum_{k=0}^{N} \gamma a_k (E_1^-)^k, \tag{23}$$

and it is equivalent to multiplication of the generating series by γ followed by truncation.

Taking into account the property (7) of the matrix E_1^-, we obtain the product of A_N and B_N:

$$A_N B_N = \left(\sum_{k=0}^{N} a_k (E_1^-)^k\right)\left(\sum_{k=0}^{N} b_k (E_1^-)^k\right) \tag{24}$$

$$= \sum_{k=0}^{N} \left(\sum_{i=0}^{k} a_i b_{k-i}\right)(E_1^-)^k. \tag{25}$$

Using the truncation operation, the product of the matrices A_N and B_N can also be expressed in terms of their generating series:

$$\begin{aligned} A_N B_N &\longleftrightarrow \operatorname{trunc}_N(\lambda(z)\mu(z)) = \operatorname{trunc}_N\left(\sum_{k=0}^{\infty}\left(\sum_{i=0}^{k} a_i b_{k-i}\right) z^k\right) \\ &= \sum_{k=0}^{N}\left(\sum_{i=0}^{k} a_i b_{k-i}\right) z^k \longleftrightarrow \sum_{k=0}^{N}\left(\sum_{i=0}^{k} a_i b_{k-i}\right)(E_1^-)^k. \end{aligned} \tag{26}$$

In other words, the product of two lower triangular matrices A_N and B_N is equivalent to the truncated product of their generating series.

The use of the generating series is especially convenient for inverting the lower triangular strip matrices.

If A_N is a lower triangular strip matrix with a generating function $\lambda(z)$, then the generating function for the inverse matrix $(A_N)^{-1}$ is simply $\mathrm{y}(z) = \lambda^{-1}(z)$. Indeed,

$$A_N\,(A_N)^{-1} \longleftrightarrow \mathrm{trunc}_N\left(\lambda(z)\,\lambda^{-1}(z)\right) = 1 \longleftrightarrow E. \tag{27}$$

This means that the coefficients on the first column of the inverse matrix $(A_N)^{-1}$ are the coefficients of the polynomial

$$\mathrm{y}_N(z) = \mathrm{trunc}_N\left(\lambda^{-1}(z)\right), \tag{28}$$

which is the truncation of the generating series for the inverse matrix. This method of inversion of triangular strip matrices is even simpler than the formulas (10) and (11).

All the above rules involving generating functions can also be used for upper triangular strip matrices.

4. Integer-order differentiation

Let us consider equidistant nodes with the step h: $t_k = kh$, $(k = 0, \dots, N)$, in the interval $[a, b]$, where $t_0 = a$ and $t_N = b$.

4.1. Backward differences

For a function $f(t)$, differentiable in $[a, b]$, we can consider first-order approximation of its derivative $f'(t)$ at the points t_k, $k = 1, \dots, N$, using first-order backward differences:

$$f'(t_k) \approx \frac{1}{h}\,\nabla f(t_k) = \frac{1}{h}\,(f_k - f_{k-1}), \quad k = 1, \dots, N. \tag{29}$$

All the N formulas (29) can be written simultaneously in the matrix form:

$$\begin{bmatrix} h^{-1}\,f_0 \\ h^{-1}\,\nabla f(t_1) \\ h^{-1}\,\nabla f(t_2) \\ \vdots \\ h^{-1}\,\nabla f(t_{N-1}) \\ h^{-1}\,\nabla f(t_N) \end{bmatrix} = \frac{1}{h} \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 \\ \dots & \dots & \dots & \ddots & \dots & \dots \\ 0 & \cdots & 0 & -1 & 1 & 0 \\ 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{N-1} \\ f_N \end{bmatrix}. \tag{30}$$

In the formula (30) the column vector of function values f_k $(k = 0, \dots, N)$ is multiplied by the matrix

$$B_N^1 = \frac{1}{h} \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 \\ \dots & \dots & \dots & \ddots & \dots & \dots \\ 0 & \cdots & 0 & -1 & 1 & 0 \\ 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}, \tag{31}$$

and the result is the column vector of approximated values of $f'(t_k)$, $k = 1, \ldots, N$, with the exception of the first element, depending on the value of the function $f(t)$ at the initial point, namely $h^{-1} f_0 = h^{-1} f(a)$. We can look at the matrix B_N^1 as at a discrete analogue of first-order differentiation. The generating function for the matrix B_N^1 is

$$\beta_1(z) = h^{-1}(1 - z). \tag{32}$$

Similarly, we can consider the approximation of the second-order derivative using second-order backward differences:

$$f''(t_k) \approx \frac{1}{h^2}\, \nabla^2 f(t_k) = \frac{1}{h^2}\left(f_k - 2f_{k-1} + f_{k-2}\right), \quad k = 2, \ldots, N, \tag{33}$$

which in the matrix form corresponds to the relationship

$$\begin{bmatrix} h^{-2}\, f_0 \\ h^{-2}\, (-2f_0 + f_1) \\ h^{-2}\, \nabla^2 f(t_2) \\ \vdots \\ h^{-2}\, \nabla^2 f(t_{N-1}) \\ h^{-2}\, \nabla^2 f(t_N) \end{bmatrix} = \frac{1}{h^2} \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -2 & 1 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ \cdots & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & \cdots & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{N-1} \\ f_N \end{bmatrix}. \tag{34}$$

In the formula (34) the column vector of function values f_k $(k = 0, \ldots, N)$ is multiplied by the matrix

$$B_N^2 = \frac{1}{h^2} \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -2 & 1 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ \cdots & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & \cdots & 1 & -2 & 1 \end{bmatrix} \tag{35}$$

and the result is the column vector of approximations of $f''(t_k)$, $k = 2, 3, \ldots, N$, with the exception of the first two elements, namely $h^{-2} f_0$ and $h^{-2}(-2f_0 + f_1)$. We can look at the matrix B_N^2 as at a discrete analogue of second-order differentiation. The generating function for the matrix B_N^2 is

$$\beta_2(z) = h^{-2}(1 - 2z + z^2) = h^{-2}(1 - z)^2. \tag{36}$$

Further, we can consider a matrix B_N^p, where p is a positive integer:

$$B_N^p = \frac{1}{h^p} \begin{bmatrix} \omega_0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\ \omega_1 & \omega_0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ \omega_2 & \omega_1 & \omega_0 & 0 & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ddots & \ldots & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ddots & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ddots & 0 & \ldots \\ 0 & \ldots & 0 & \omega_p & \omega_{p-1} & \ldots & \omega_0 & 0 \\ 0 & 0 & \ldots & 0 & \omega_p & \omega_{p-1} & \ldots & \omega_0 \end{bmatrix}, \tag{37}$$

$$\omega_j = (-1)^j \binom{p}{j}, \qquad j = 0, 1, 2, \ldots, p. \tag{38}$$

The matrix B_N^p is a discrete analogue of differentiation of p-th order, if backward differences of the p-th order are used. The generating function for the matrix B_N^p is

$$\beta_p(z) = h^{-p}(1-z)^p. \tag{39}$$

For the generating functions of the form $\beta_p(z)$ we have:

$$\begin{aligned} \beta_2(z) &= \beta_1(z)\beta_1(z) \\ \beta_p(z) &= \underbrace{\beta_1(z)\ldots\beta_1(z)}_{p} \\ \beta_{p+q}(z) &= \beta_p(z)\beta_q(z) = \beta_q(z)\beta_p(z), \end{aligned}$$

from which in view of (26) follows that

$$B_N^2 = B_N^1 \, B_N^1, \tag{40}$$

$$B_N^p = \underbrace{B_N^1 \, B_N^1 \ldots B_N^1}_{p}, \tag{41}$$

$$B_N^{p+q} = B_N^p \, B_N^q = B_N^q \, B_N^p, \tag{42}$$

where p and q are positive integers.

4.2. Forward differences

Similarly to the previous section, we obtain that the matrix F_N^p, where p is a positive integer,

$$F_N^p = \frac{1}{h^p} \begin{bmatrix} \omega_0 & \dots & \omega_{p-1} & \omega_p & 0 & \dots & 0 & 0 \\ 0 & \omega_0 & \dots & \omega_{p-1} & \omega_p & 0 & \dots & 0 \\ \dots & 0 & \ddots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \ddots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & \ddots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & \omega_0 & \omega_1 & \omega_2 \\ 0 & 0 & 0 & 0 & \dots & 0 & \omega_0 & \omega_1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & \omega_0 \end{bmatrix}, \tag{43}$$

$$\omega_j = (-1)^j \binom{p}{j}, \qquad j = 0, 1, 2, \dots, p. \tag{44}$$

is a discrete analogue of differentiation of p-th order, namely of $(-1)^p f^{(p)}(t)$, if forward differences of the p-th order are used. The generating function for F_N^p is the same as for B_N^p: $\beta_p(z) = h^{-p}(1-z)^p$.

Since the generating functions are the same as in case of the matrices B_N^p, we have for F_N^p the similar properties:

$$F_N^2 = F_N^1 \, F_N^1, \tag{45}$$

$$F_N^p = \underbrace{F_N^1 \, F_N^1 \dots F_N^1}_{p}, \tag{46}$$

$$F_N^{p+q} = F_N^p \, F_N^q = F_N^q \, F_N^p, \tag{47}$$

where p and q are positive integers.

It also should be noted that transposition of the matrix B_N^p, representing the backward difference operation, gives the matrix F_N^p, which corresponds to forward differentiating:

$$\left(B_N^p\right)^T = F_N^p, \qquad \left(F_N^p\right)^T = B_N^p. \tag{48}$$

5. n-fold integration

Now let us turn to the integration. To deal with operations, which are inverse to the differentiation, we have to consider definite integrals with one limit fixed and another moving.

5.1. Moving upper limit of integration

Let us take a function $f(t)$, integrable in $[a, b]$, and consider integrals with fixed lower limit and moving upper limit:

$$g_1(t) = \int_a^t f(t)dt, \tag{49}$$

for which we have $g_1'(t) = f(t)$ in (a,b).

Let us consider equidistant nodes with the step h: $t_k = kh$, $(k = 0, \ldots, N)$, in the interval $[a,b]$, where $t_0 = a$ and $t_N = b$. We can use the left rectangular quadrature rule for approximating the integral (49) at the points t_k, $k = 1, \ldots, N$:

$$g_1(t_k) \approx h \sum_{i=0}^{k-1} f_i, \qquad k = 1, \ldots, N. \tag{50}$$

All the N formulas (50) can be written simultaneously in the matrix form:

$$\begin{bmatrix} g_1(t_1) \\ g_1(t_2) \\ g_1(t_3) \\ \vdots \\ g_1(t_N) \\ g_1(t_N + h) \end{bmatrix} = h \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ 1 & \cdots & 1 & 1 & 1 & 0 \\ 1 & 1 & \cdots & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{N-1} \\ f_N \end{bmatrix}. \tag{51}$$

We see that the column vector of function values f_k $(k = 0, \ldots, N)$ is multiplied by the matrix

$$I_N^1 = h \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ 1 & \cdots & 1 & 1 & 1 & 0 \\ 1 & 1 & \cdots & 1 & 1 & 1 \end{bmatrix}, \tag{52}$$

and the result is the column vector of approximated values of the integral (49), namely $g_1(t_k)$, $k = 1, \ldots, N$, with the exception of the last element, which corresponds to the node lying outside of the considered interval $[a,b]$. We can look at the matrix I_N^1 as at a discrete analogue of left rectangular quadrature rule for evaluating the integral (49). The generating function for I_N^1 is

$$\varphi_1(z) = h(1-z)^{-1}. \tag{53}$$

It must be noted here that the matrix I_N^1 is inverse to the matrix B_N^1, which corresponds to backward difference approximation of the first derivative. We have:

$$B_N^1 I_N^1 = I_N^1 B_N^1 \longleftrightarrow \mathrm{trunc}_N \left(\beta_1(z)\,\varphi_1(z)\right) = 1 \longleftrightarrow E. \tag{54}$$

Therefore, having one of these matrices, we can immediately obtain another by a matrix inversion.

Similarly, we can consider the two-fold integral with a moving upper boundary:

$$g_2(t) = \int\limits_a^t dt \int\limits_a^t f(t)dt, \tag{55}$$

for which we have $g_2''(t) = g_1'(t) = f(t)$ in (a, b).

Using the left rectangular quadrature rule twice for approximating $g_2(t_k)$ and taking into account that $g_1(t_0) = 0$, we have:

$$\begin{aligned} g_2(t_k) &= h\sum_{i=0}^{k-1} g_1(t_i) = h\sum_{i=1}^{k-1} g_1(t_i) = h\sum_{i=1}^{k-1} h\sum_{j=0}^{i-1} f_j \\ &= h^2 \sum_{i=1}^{k-1}\sum_{j=0}^{i-1} f_j = h^2 \sum_{j=0}^{k-2}(k-j-1)\, f_j \\ &= h^2\Big((k-1)f_0 + (k-2)f_1 + \ldots + 2f_{k-3} + f_{k-2}\Big), \\ &\qquad k = 2, 3, \ldots, N. \end{aligned} \tag{56}$$

The equations (56) can be written simultaneously in the matrix form:

$$\begin{bmatrix} g_2(t_2) \\ g_2(t_3) \\ \vdots \\ g_2(t_N) \\ g_2(t_N+h) \\ g_2(t_N+2h) \end{bmatrix} = h^2 \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 2 & 1 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots & \cdots & \cdots \\ \cdots & 3 & 2 & 1 & 0 & 0 \\ N & \cdots & 3 & 2 & 1 & 0 \\ N+1 & N & \cdots & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{N-2} \\ f_{N-1} \\ f_N \end{bmatrix}. \tag{57}$$

We see that the column vector of function values f_k $(k = 0, \ldots, N)$ is multiplied by the matrix

$$I_N^2 = h^2 \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 2 & 1 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots & \cdots & \cdots \\ \cdots & 3 & 2 & 1 & 0 & 0 \\ N & \cdots & 3 & 2 & 1 & 0 \\ N+1 & N & \cdots & 3 & 2 & 1 \end{bmatrix}, \tag{58}$$

and the result is the column vector of approximated values of the integral (55), namely $g_2(t_k)$, $k = 2, \ldots, N$, with the exception of the last two elements, which correspond to the nodes lying outside of the considered interval $[a, b]$. We can look at the matrix I_N^2 as at a discrete analogue of left rectangular quadrature rule for evaluating the two-fold integral (55). The generating function for I_N^2 is

$$\varphi_2(z) = h^2(1-z)^{-2}. \tag{59}$$

It must be mentioned here that the matrix I_N^2 is inverse to the matrix B_N^2, which corresponds to backward difference approximation of the second derivative. We have:

$$B_N^2 I_N^2 = I_N^2 B_N^2 \longleftrightarrow \mathrm{trunc}_N\,(\beta_2(z)\,\varphi_2(z)) = 1 \longleftrightarrow E. \tag{60}$$

Therefore, having one of these matrices, we can immediately obtain another by a matrix inversion.

If we consider p-fold integration with a moving upper limit,

$$g_p(t) = \int_a^t d\tau_p \int_a^{\tau_p} d\tau_{p-1} \dots \int_a^{\tau_2} f(\tau_1) d\tau_1, \tag{61}$$

and apply the left rectangular quadrature rule p times, then we arrive at the following relationship in the matrix form:

$$\begin{bmatrix} g_p(t_p) \\ g_p(t_{p+1}) \\ \vdots \\ g_p(t_N) \\ \vdots \\ g_p(t_N+h) \\ g_p(t_N+ph) \end{bmatrix} = h^p \begin{bmatrix} \gamma_0 & 0 & 0 & 0 & \cdots & \cdots & 0 \\ \gamma_1 & \gamma_0 & 0 & 0 & \cdots & \cdots & 0 \\ \dots & \dots & \ddots & \dots & \dots & \dots & \dots \\ \cdots & \gamma_2 & \gamma_1 & \gamma_0 & 0 & \cdots & \cdots \\ \dots & \dots & \dots & \dots & \ddots & \dots & \dots \\ \gamma_{N-1} & \cdots & \cdots & \gamma_2 & \gamma_1 & \gamma_0 & 0 \\ \gamma_N & \gamma_{N-1} & \cdots & \cdots & \gamma_2 & \gamma_1 & \gamma_0 \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_p \\ \vdots \\ f_{N-1} \\ f_N \end{bmatrix}, \tag{62}$$

involving the lower triangular strip matrix I_N^p with the generating function $\varphi_p(z) = h^p(1-z)^{-p}$,

$$I_N^p = h^p \begin{bmatrix} \gamma_0 & 0 & 0 & 0 & \cdots & \cdots & 0 \\ \gamma_1 & \gamma_0 & 0 & 0 & \cdots & \cdots & 0 \\ \dots & \dots & \ddots & \dots & \dots & \dots & \dots \\ \cdots & \gamma_2 & \gamma_1 & \gamma_0 & 0 & \cdots & \cdots \\ \dots & \dots & \dots & \dots & \ddots & \dots & \dots \\ \gamma_{N-1} & \cdots & \cdots & \gamma_2 & \gamma_1 & \gamma_0 & 0 \\ \gamma_N & \gamma_{N-1} & \cdots & \cdots & \gamma_2 & \gamma_1 & \gamma_0 \end{bmatrix}, \tag{63}$$

which is inverse to the matrix B_N^p, corresponding to the backward difference approximation of the p-th derivative:

$$B_N^p I_N^p = I_N^p B_N^p \longleftrightarrow \mathrm{trunc}_N\,(\beta_p(z)\,\varphi_p(z)) = 1 \longleftrightarrow E. \tag{64}$$

In view of (26) it follows from the properties of the generating functions $\varphi_p(z) = h^p(1-z)^{-p}$ that

$$I_N^2 = I_N^1\, I_N^1, \tag{65}$$
$$I_N^p = \underbrace{I_N^1\, I_N^1 \ldots I_N^1}_{p}, \tag{66}$$
$$I_N^{p+q} = I_N^p\, I_N^q = I_N^q\, I_N^p, \tag{67}$$

where p and q are positive integers. Moreover, the matrices I_N^p commute also with the matrices B_N^p.

5.2. Moving lower limit of integration

If we consider p-fold integration with a moving lower limit,

$$y_p(t) = \int\limits_t^b d\tau_p \int\limits_{\tau_p}^b d\tau_{p-1} \ldots \int\limits_{\tau_2}^b f(\tau_1) d\tau_1, \tag{68}$$

then its discrete analogue is represented by the upper triangular strip matrix J_N^p with the generating function $\varphi_p(z) = h^p(1-z)^{-p}$:

$$J_N^p = h^p \begin{bmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \cdots & \gamma_{N-1} & \gamma_N \\ 0 & \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \cdots & \gamma_{N-1} \\ \ldots & \ldots & \ddots & \ldots & \ldots & \ldots & \ldots \\ \cdots & 0 & 0 & \gamma_0 & \gamma_1 & \gamma_2 & \cdots \\ \ldots & \ldots & \ldots & \ldots & \ddots & \ldots & \ldots \\ 0 & \cdots & \cdots & 0 & 0 & \gamma_0 & \gamma_1 \\ 0 & 0 & \cdots & \cdots & 0 & 0 & \gamma_0 \end{bmatrix}. \tag{69}$$

The matrix J_N^p is inverse to the matrix F_N^p, corresponding to the backward difference approximation of the p-th derivative:

$$F_N^p J_N^p = J_N^p F_N^p \longleftrightarrow \mathrm{trunc}_N\,(\beta_p(z)\,\varphi_p(z)) = 1 \longleftrightarrow E. \tag{70}$$

In view of (26) it follows from the properties of the generating functions $\varphi_p(z) = h^p(1-z)^{-p}$ that

$$J_N^2 = J_N^1\, J_N^1, \tag{71}$$
$$J_N^p = \underbrace{J_N^1\, J_N^1 \ldots J_N^1}_{p}, \tag{72}$$
$$J_N^{p+q} = J_N^p\, J_N^q = J_N^q\, J_N^p, \tag{73}$$

where p and q are positive integers. Moreover, the matrices J_N^p commute also with the matrices F_N^p.

It also should be noted that transposition of the matrix I_N^p, representing the integration with moving upper limit, gives the matrix J_N^p, which corresponds to integration with moving lower limit:

$$\left(I_N^p\right)^T = J_N^p, \qquad \left(J_N^p\right)^T = I_N^p. \tag{74}$$

6. Fractional differentiation

The triangular strip matrices can also be used for fractional derivatives. In this case, we arrive at lower (upper) triangular matrices, which have no zeros below (above) the main diagonal.

6.1. Left-sided fractional derivatives

Let us consider a function $f(t)$, defined in $[a,b]$, such that $f(t) \equiv 0$ for $t < a$. (Functions satisfying this condition are often called causal functions.) We assume that the function $f(t)$ is good enough for considering its left-sided fractional derivative of real order α $(n-1 \leq \alpha < n)$,

$${}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t \frac{f(\tau)d\tau}{(t-\tau)^{\alpha-n+1}}, \qquad (a < t < b). \tag{75}$$

Let us take equidistant nodes with the step h: $t_k = kh$ $(k = 0, 1, \ldots, N)$, in the interval $[a,b]$, where $t_0 = a$ and $t_N = b$. Using the backward fractional difference approximation for the α-th derivative at the points t_k, $k = 0, 1, \ldots, N$, we have:

$${}_aD_{t_k}^\alpha f(t) \approx \frac{\nabla^\alpha f(t_k)}{h^\alpha} = h^{-\alpha}\sum_{j=0}^{k}(-1)^j\binom{\alpha}{j} f_{k-j}, \qquad k = 0, 1, \ldots, N. \tag{76}$$

All $N+1$ formulas (76) can be written simultaneously in the matrix form:

$$\begin{bmatrix} h^{-\alpha}\nabla^\alpha f(t_0) \\ h^{-\alpha}\nabla^\alpha f(t_1) \\ h^{-\alpha}\nabla^\alpha f(t_2) \\ \vdots \\ h^{-\alpha}\nabla^\alpha f(t_{N-1}) \\ h^{-\alpha}\nabla^\alpha f(t_N) \end{bmatrix} = \frac{1}{h^\alpha}\begin{bmatrix} \omega_0^{(\alpha)} & 0 & 0 & 0 & \cdots & 0 \\ \omega_1^{(\alpha)} & \omega_0^{(\alpha)} & 0 & 0 & \cdots & 0 \\ \omega_2^{(\alpha)} & \omega_1^{(\alpha)} & \omega_0^{(\alpha)} & 0 & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \cdots & \cdots \\ \omega_{N-1}^{(\alpha)} & \ddots & \omega_2^{(\alpha)} & \omega_1^{(\alpha)} & \omega_0^{(\alpha)} & 0 \\ \omega_N^{(\alpha)} & \omega_{N-1}^{(\alpha)} & \ddots & \omega_2^{(\alpha)} & \omega_1^{(\alpha)} & \omega_0^{(\alpha)} \end{bmatrix}\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{N-1} \\ f_N \end{bmatrix}, \tag{77}$$

$$\omega_j^{(\alpha)} = (-1)^j\binom{\alpha}{j}, \qquad j = 0, 1, \ldots, N. \tag{78}$$

In the formula (77) the column vector of function values f_k $(k = 0, \dots, N)$ is multiplied by the matrix

$$B_N^{\alpha} = \frac{1}{h^{\alpha}} \begin{bmatrix} \omega_0^{(\alpha)} & 0 & 0 & 0 & \cdots & 0 \\ \omega_1^{(\alpha)} & \omega_0^{(\alpha)} & 0 & 0 & \cdots & 0 \\ \omega_2^{(\alpha)} & \omega_1^{(\alpha)} & \omega_0^{(\alpha)} & 0 & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \cdots & \cdots \\ \omega_{N-1}^{(\alpha)} & \ddots & \omega_2^{(\alpha)} & \omega_1^{(\alpha)} & \omega_0^{(\alpha)} & 0 \\ \omega_N^{(\alpha)} & \omega_{N-1}^{(\alpha)} & \ddots & \omega_2^{(\alpha)} & \omega_1^{(\alpha)} & \omega_0^{(\alpha)} \end{bmatrix} \tag{79}$$

and the result is the column vector of approximated values of the fractional derivative ${}_aD_{t_k}^{\alpha} f(t)$, $k = 0, 1, \dots, N$. We can look at the matrix B_N^{α} as at a discrete analogue of left-sided fractional differentiation of order α.

The generating function for the matrix B_N^{α} is

$$\beta_{\alpha}(z) = h^{-\alpha}(1 - z)^{\alpha}. \tag{80}$$

Since for lower triangular matrices B_N^{α} and B_N^{β} we always have

$$B_N^{\alpha} B_N^{\beta} = B_N^{\beta} B_N^{\alpha} = B_N^{\alpha+\beta},$$

we can consider such matrices as discrete analogues of the corresponding left-sided fractional derivatives ${}_aD_t^{\alpha}$ and ${}_aD_t^{\beta}$, where $n - 1 \leq \alpha < n$ and $m - 1 \leq \alpha < m$, only if

$${}_aD_t^{\alpha}\Big({}_aD_t^{\beta} f(t)\Big) = {}_aD_t^{\beta}\Big({}_aD_t^{\alpha} f(t)\Big) = {}_aD_t^{\alpha+\beta} f(t),$$

which holds if

$$f^{(k)}(a) = 0, \quad k = 1, 2, \dots, r - 1, \tag{81}$$

where $r = \max\{n, m\}$.

This means that if left-sided fractional derivatives of a function $f(t)$ of orders less than some integer r are considered, than they can all be replaced with their corresponding discrete analogues, if the function $f(t)$ satisfies the conditions (81).

6.2. Right-sided fractional derivatives

Let us consider a function $f(t)$, defined in $[a, b]$, such that $f(t) \equiv 0$ for $t > b$. We assume that the function $f(t)$ is good enough for considering its right-sided fractional derivative of real order α $(n - 1 \leq \alpha < n)$,

$${}_tD_b^{\alpha} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_t^b \frac{f(\tau)d\tau}{(\tau - t)^{\alpha - n + 1}}, \qquad (a < t < b). \tag{82}$$

Similarly to the previous section, we can obtain the discrete analogue of the right-sided fractional differentiation on the net of equidistant nodes with the step h: $t_k = kh$ $(k = 0, 1, \ldots, N)$, in the interval $[a, b]$, where $t_0 = a$ and $t_N = b$, which is represented by the matrix

$$F_N^{\alpha} = \frac{1}{h^{\alpha}} \begin{bmatrix} \omega_0^{(\alpha)} & \omega_1^{(\alpha)} & \ddots & \ddots & \omega_{N-1}^{(\alpha)} & \omega_N^{(\alpha)} \\ 0 & \omega_0^{(\alpha)} & \omega_1^{(\alpha)} & \ddots & \ddots & \omega_{N-1}^{(\alpha)} \\ 0 & 0 & \omega_0^{(\alpha)} & \omega_1^{(\alpha)} & \ddots & \ddots \\ \cdots & \cdots & \cdots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & 0 & \omega_0^{(\alpha)} & \omega_1^{(\alpha)} \\ 0 & 0 & \cdots & 0 & 0 & \omega_0^{(\alpha)} \end{bmatrix}. \qquad (83)$$

The generating function for the matrix F_N^{α} is the same as for the matrix B_N^{α}: $\beta_{\alpha}(z) = h^{-\alpha}(1-z)^{\alpha}$.

It should also be mentioned that transposition of the matrix B_N^{α}, corresponding to left-sided fractional differentiation, gives the matrix F_N^{p}, which corresponds to right-sided differentiation:

$$\left(B_N^{\alpha}\right)^T = F_N^{\alpha}, \qquad \left(F_N^{\alpha}\right)^T = B_N^{\alpha}. \qquad (84)$$

Similarly to the previous section, if right-sided fractional derivatives of a function $f(t)$ of orders less than some integer r are considered, then they can all be replaced with their corresponding discrete analogues, if the function $f(t)$ satisfies the conditions

$$f^{(k)}(b) = 0, \quad k = 1,\, 2,\, \ldots\, , r-1. \qquad (85)$$

6.3. Sequential fractional derivatives

For left-sided sequential fractional derivatives, in which all derivatives in the sequence can be arbitrarily interchanged,

$$ {}_a\mathcal{D}_t^{\vec{\alpha}} f(t) = {}_aD_t^{\alpha_1}\, {}_aD_t^{\alpha_2} \, \ldots \, {}_aD_t^{\alpha_n} f(t), \qquad (86)$$

$$\vec{\alpha} = (\alpha_1,\, \alpha_2,\, \ldots,\, \alpha_n),$$

(and the same equidistant nodes as above) the discrete analogue $\mathcal{B}_N^{\vec{\alpha}}$ has the form of the product of matrices $B_N^{\alpha_k}$, corresponding to operators ${}_aD_t^{\alpha_k}$, $k = 1, 2, \ldots, n$:

$$\mathcal{B}_N^{\vec{\alpha}} = B_N^{\alpha_1} B_N^{\alpha_2} \ldots B_N^{\alpha_n} = \prod_{k=1}^{n} B_N^{\alpha_k}. \qquad (87)$$

Similarly, for right-sided fractional derivatives, in which all derivatives in the sequence can be arbitrarily interchanged,

$$ {}_t\mathcal{D}_b^{\vec{\alpha}} f(t) = {}_tD_b^{\alpha_1}\, {}_tD_b^{\alpha_2} \, \ldots \, {}_tD_b^{\alpha_n} f(t), \qquad (88)$$

the discrete analogue $\mathcal{F}_N^{\vec{\alpha}}$ is

$$\mathcal{F}_N^{\vec{\alpha}} = F_N^{\alpha_1} F_N^{\alpha_2} \ldots F_N^{\alpha_n} = \prod_{k=1}^{n} F_N^{\alpha_k}. \tag{89}$$

7. Fractional integration

Discrete analogues of left- and right-sided fractional integrals can be obtained by inversion of the discrete analogues of the corresponding fractional derivatives.

7.1. Left-sided fractional integration

To obtain the matrix I_N^{α}, corresponding to the discrete analogue of the left-sided fractional integration ($\alpha > 0$),

$${}_aD_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int\limits_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \qquad (a < t < b), \tag{90}$$

we simply invert the matrix B_N^{α}, corresponding to the left-sided fractional differentiation:

$$I_N^{\alpha} = \left(B_N^{\alpha}\right)^{-1}. \tag{91}$$

If $\varphi_\alpha(z)$ denotes the generation function for I_N^{α} and $\beta_\alpha(z)$ is the generating function for B_N^{α}, then taking into account the rule (28), we can write:

$$I_N^{\alpha} \longleftrightarrow \varphi_N(z) = \mathrm{trunc}_N \left(\beta_\alpha^{-1}(z)\right) = \mathrm{trunc}_N \left(h^{\alpha}(1-z)^{-\alpha}\right).$$

Therefore, the matrix I_N^{α} has the following form:

$$I_N^{\alpha} = h^{\alpha} \begin{bmatrix} \omega_0^{(-\alpha)} & 0 & 0 & 0 & \cdots & 0 \\ \omega_1^{(-\alpha)} & \omega_0^{(-\alpha)} & 0 & 0 & \cdots & 0 \\ \omega_2^{(-\alpha)} & \omega_1^{(-\alpha)} & \omega_0^{(-\alpha)} & 0 & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \cdots & \cdots \\ \omega_{N-1}^{(-\alpha)} & \ddots & \omega_2^{(-\alpha)} & \omega_1^{(-\alpha)} & \omega_0^{(-\alpha)} & 0 \\ \omega_N^{(-\alpha)} & \omega_{N-1}^{(-\alpha)} & \ddots & \omega_2^{(-\alpha)} & \omega_1^{(-\alpha)} & \omega_0^{(-\alpha)} \end{bmatrix}. \tag{92}$$

7.2. Right-sided fractional integration

Similarly, inversion of the matrix F_N^{α}, corresponding to the right-sided fractional differentiation, gives the matrix J_N^{α},

$$J_N^\alpha = h^\alpha \begin{bmatrix} \omega_0^{(-\alpha)} & \omega_1^{(-\alpha)} & \ddots & \ddots & \omega_{N-1}^{(-\alpha)} & \omega_N^{(-\alpha)} \\ 0 & \omega_0^{(-\alpha)} & \omega_1^{(-\alpha)} & \ddots & \ddots & \omega_{N-1}^{(-\alpha)} \\ 0 & 0 & \omega_0^{(-\alpha)} & \omega_1^{(-\alpha)} & \ddots & \ddots \\ \cdots & \cdots & \cdots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & 0 & \omega_0^{(-\alpha)} & \omega_1^{(-\alpha)} \\ 0 & 0 & \cdots & 0 & 0 & \omega_0^{(-\alpha)} \end{bmatrix} \tag{93}$$

which is the discrete analogue of the right-sided fractional integration ($\alpha > 0$):

$${}_tD_b^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau - t)^{\alpha-1} f(\tau) d\tau, \qquad (a < t < b). \tag{94}$$

7.3. Feller and Riesz potentials on a finite interval

Let us consider the Riesz potential $R^\alpha f(t)$ and the modified Riesz potential $M^\alpha f(t)$ on a finite interval [13, Chap. 3]:

$$R^\alpha f(t) = \frac{1}{2\Gamma(\alpha)\cos(\alpha\pi/2)} \int_a^b \frac{\varphi(\tau) d\tau}{|t-\tau|^{1-\alpha}}, \qquad (a < t < b), \tag{95}$$

$$M^\alpha f(t) = \frac{1}{2\Gamma(\alpha)\sin(\alpha\pi/2)} \int_a^b \frac{\operatorname{sign}(t-\tau)\varphi(\tau) d\tau}{|t-\tau|^{1-\alpha}}, \qquad (a < t < b). \tag{96}$$

Obviously, both these operators are linear combinations of the left-sided and right-sided fractional integrals:

$$R^\alpha f(t) = \frac{1}{2\cos(\alpha\pi/2)} \left({}_aD_t^{-\alpha} f(t) + {}_tD_b^{-\alpha} f(t) \right), \tag{97}$$

$$M^\alpha f(t) = \frac{1}{2\sin(\alpha\pi/2)} \left({}_aD_t^{-\alpha} f(t) - {}_tD_b^{-\alpha} f(t) \right). \tag{98}$$

The matrices $\mathcal{R}_N^\alpha$ and $\mathcal{M}_N^\alpha$ of their discrete analogues are linear combinations of the matrix I_N^α, corresponding to the left-sided fractional integral, and the matrix J_N^α, corresponding to the right-sided fractional integral:

$$\mathcal{R}_N^\alpha = \frac{1}{2\cos(\alpha\pi/2)} \left(I_N^\alpha + J_N^\alpha \right), \tag{99}$$

$$\mathcal{M}_N^\alpha = \frac{1}{2\sin(\alpha\pi/2)} \left(I_N^\alpha - J_N^\alpha \right). \tag{100}$$

The Feller potential operator $\Phi^\alpha f(t)$ is also a linear combination of left- and right-sided fractional integrals, but with general constant coefficients u, v [13, Ch.3]:

$$\Phi^{\alpha} f(t) = u\, {}_a D_t^{-\alpha} + v\, {}_t D_b^{-\alpha} f(t), \tag{101}$$

and the matrix of its discrete analogue is

$$\Phi_N^{\alpha} = u I_N^{\alpha} + v J_N^{\alpha}. \tag{102}$$

Numerical inversion of the Riesz and Feller potential operators (95), (96), and (101), reduces to inversion of the corresponding square matrices $\mathcal{R}_N^{\alpha}$, $\mathcal{M}_N^{\alpha}$, and Φ_N^{α}.

Example 1. Let us consider the fractional integral equation with the Riesz kernel:

$$\frac{1}{\Gamma(1-\alpha)} \int_{-1}^{1} \frac{y(\tau)\, d\tau}{|t-\tau|^{\alpha}} = 1, \quad (-1 < t < 1), \tag{103}$$

which has the solution [12, eq. 6.116]

$$y(t) = \pi^{-1} \Gamma(1-\alpha) \cos\left(\frac{\alpha\pi}{2}\right) (1-t^2)^{(\alpha-1)/2}. \tag{104}$$

Writing the equation (103) in the form

$${}_{-1}D_t^{-(1-\alpha)} y(t) + {}_t D_1^{-(1-\alpha)} y(t) = 1,$$

and replacing the fractional derivatives with their discrete analogues, we obtain the system of linear algebraic equations

$$\left(B_N^{-(1-\alpha)} + F_N^{-(1-\alpha)}\right) Y_N = F_N, \tag{105}$$

where $F_N = \left(1, 1, \ldots, 1\right)^T$, for determination of $Y_N = \left(y(t_0), y(t_1), \ldots, y(t_N)\right)^T$, which represents the approximate solution of the equations (103).

The numerical solution of the equation (103) for $\alpha = 0.8$ is shown in **Fig. 1**. In this figure, and also in all subsequent figures, only a subset of the points of the obtained numerical solution is shown; otherwise it would be difficult to depict the analytical solution and the numerical one, which are very close each other.

8. Numerical solution of fractional differential equations

The use of triangular strip matrices significantly simplifies numerical solution of fractional differential equations. Instead of writing unwieldy recurrence relationships for determination of the values of the unknown function in equidistant discretization nodes, one can immediately write a system of algebraic equations for those values.

For convenience, let us introduce a certain type of matrices, which are obtained from the $N \times N$ unit matrix E by keeping only some of its rows and omitting all other rows: S_1 is obtained by omitting only the first row of E; S_2 is obtained by

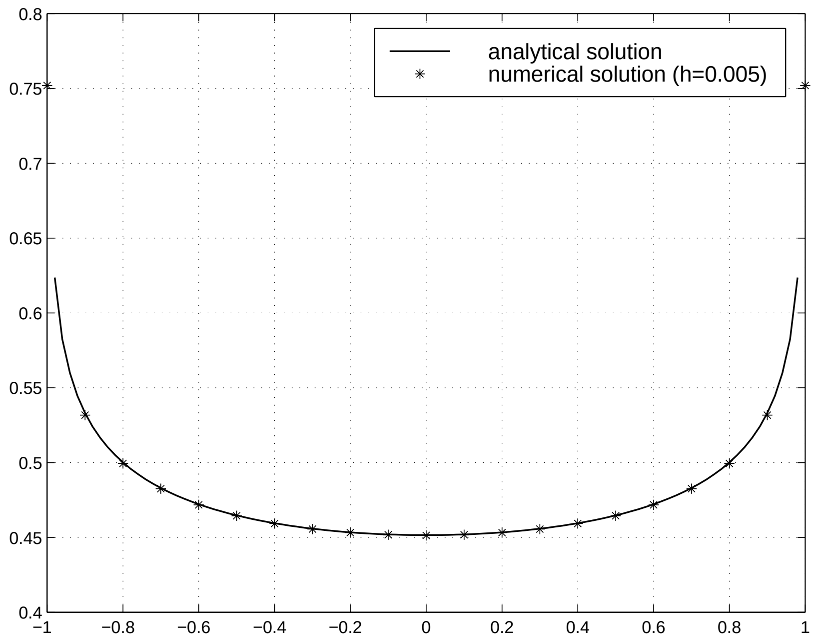

Figure 1: Solution of equation $\frac{1}{\Gamma(1-\alpha)} \int_{-1}^{1} |t-\tau|^{-\alpha} y(\tau)\, d\tau = 1$.

omitting only the second row; $S_{1,2}$ is obtained by omitting only the first and the second row of E; and, in general, $S_{r_1,r_2,\ldots,r_k}$ is obtained by omitting the rows with the numbers $r_1, r_2, \ldots, r_k$.

If A is a square $N \times N$ matrix, then the product $S_{r_1,r_2,\ldots,r_k}A$ contains only rows of A with the numbers different from $r_1, r_2, \ldots, r_k$. Similarly, the product $AS^T_{r_1,r_2,\ldots,r_k}$ contains only columns of A with the numbers different from $r_1, r_2, \ldots, r_k$. Because of this property, the matrix $S_{r_1,r_2,\ldots,r_k}$ is called an *eliminator*. In case of infinite matrices, similar matrices appeared in [4].

The following simple example illustrates the main property of eliminators:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}; \qquad S_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \qquad S_1\, A = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix};$$

$$A\, S_1^T = \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}; \qquad S_1\, A\, S_1^T = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}.$$

Considering $(N+1) \times (N+1)$ lower triangular matrices L_N (1) and upper triangular matrices U_N (2), and numbering rows and columns from 0 to N, we have the following useful relationships:

$$S_0 \left\{ \begin{array}{c} L_N \\ U_N \end{array} \right\} S_0^T = \left\{ \begin{array}{c} L_{N-1} \\ U_{N-1} \end{array} \right\}, \tag{106}$$

$$S_N \left\{ \begin{array}{c} L_N \\ U_N \end{array} \right\} S_N^T = \left\{ \begin{array}{c} L_{N-1} \\ U_{N-1} \end{array} \right\}, \tag{107}$$

$$S_{0,1,\ldots,k} \left\{ \begin{array}{c} L_N \\ U_N \end{array} \right\} S_{0,1,\ldots,k}^T = \left\{ \begin{array}{c} L_{N-k-1} \\ U_{N-k-1} \end{array} \right\}, \tag{108}$$

$$S_{N-k,N-k+1,\ldots,N} \left\{ \begin{array}{c} L_N \\ U_N \end{array} \right\} S_{N-k,N-k+1,\ldots,N}^T = \left\{ \begin{array}{c} L_{N-k-1} \\ U_{N-k-1} \end{array} \right\}. \tag{109}$$

In other words, simultaneous multiplication of a triangular strip matrix by the eliminator $S_{0,1,\ldots,k}$ (or by $S_{N-k,N-k+1,\ldots,N}$) on the left and $S_{0,1,\ldots,k}^T$ (respectively, by $S_{N-k,N-k+1,\ldots,N}^T$) on the right preserves the type and the structure of the triangular strip matrix, and only reduces its size by $k+1$ rows and $k+1$ columns.

8.1. Initial value problems for FDEs

The general procedure of numerical solution of fractional differential equations consists of two steps.

First, initial conditions are used to reduce a given initial-value problem to a problem with zero initial conditions. At this stage, instead of a given equation a modified equation, incorporating the initial values, is obtained.

Then the system of algebraic equations is obtained by replacing all derivatives (of fractional and integer orders) in the obtained modified equation by the corresponding matrices (B_N^α for left-sided derivatives, F_N^α for right-sided derivatives) for their discrete analogues.

We will consider an m-term linear fractional differential equation with non-constant coefficients of the following form:

$$\sum_{k=1}^{m} p_k(t) D^{\alpha_k} y(t) = f(t), \tag{110}$$

$$0 \leq \alpha_1 < \alpha_2 < \ldots < \alpha_m, \qquad n-1 < \alpha_m < n,$$

where D^{α_k} denotes either Riemann-Liouville or Caputo left-sided fractional derivative of order α_k.

Let us denote

$$P_N^{(k)} = \mathrm{diag}\Big(p_k(t_0), p_k(t_1), \ldots, p_k(t_N)\Big) = \left[\begin{array}{llll} p_k(t_0) & 0 & \ldots & 0 \\ 0 & p_k(t_1) & 0 & \ldots \\ 0 & \ldots & \ddots & 0 \\ 0 & \ldots & 0 & p_k(t_N) \end{array} \right], \tag{111}$$

$$Y_N = \Big(y(t_0), y(t_1), \ldots, y(t_N)\Big)^T, \qquad F_N = \Big(f(t_0), f(t_1), \ldots, f(t_N)\Big)^T. \quad (112)$$

Using these notations and taking into account that the discrete analogue of the left-sided fractional derivative D^{α_k} is $B_N^{\alpha_k}$, we can write a discrete analogue of the fractional differential equation (110):

$$\sum_{k=1}^{m} P_N^{(k)} B_N^{\alpha_k} Y_N = F_N. \quad (113)$$

8.2. Zero initial conditions

If $n-1 < \alpha_m < n$, then the Riemann-Liouville and the Caputo formulations of the equation (110) are equivalent under the assumption of zero initial values of the function $y(t)$ and its $(n-1)$ derivatives [12]:

$$y^{(k)}(t_0) = 0, \qquad k = 0, 1, \ldots, n-1. \quad (114)$$

Approximating the derivatives in the initial conditions (114) by backward differences, we immediately obtain:

$$y(t_0) = y(t_1) = \ldots = y(t_{n-1}) = 0. \quad (115)$$

The linear algebraic system for determination of $y_n, \ldots, y_N$ is obtained from the system (113) by omitting its first n rows and substituting the zero starting values (115) into the remaining equations. This can be *symbolically* written with the help of eliminator:

$$\left\{ S_{0,1,\ldots,n-1} \left\{ \sum_{k=1}^{m} P_N^{(k)} B_N^{\alpha_k} \right\} S_{0,1,\ldots,n-1}^T \right\} \{S_{0,1,\ldots,n-1} Y_N\} = S_{0,1,\ldots,n-1} F_N. \quad (116)$$

The solution of the linear algebraic system (116) along with the starting values (115) gives the numerical solution of the fractional differential equation (110) under zero initial conditions (114).

If the coefficients $p_k(t)$ are constant, i.e. $p_k(t) \equiv p_k$, then the system (116) takes on the simplest form:

$$\sum_{k=1}^{m} p_k B_{N-n}^{\alpha_k} \{S_{0,1,\ldots,n-1} Y_N\} = S_{0,1,\ldots,n-1} F_N. \quad (117)$$

Example 2. Let us consider the following two-term fractional differential equation under zero initial conditions:

$$y^{(\alpha)}(t) + y(t) = 1, \quad (118)$$

$$y(0) = 0, \quad y'(0) = 0, \quad (119)$$

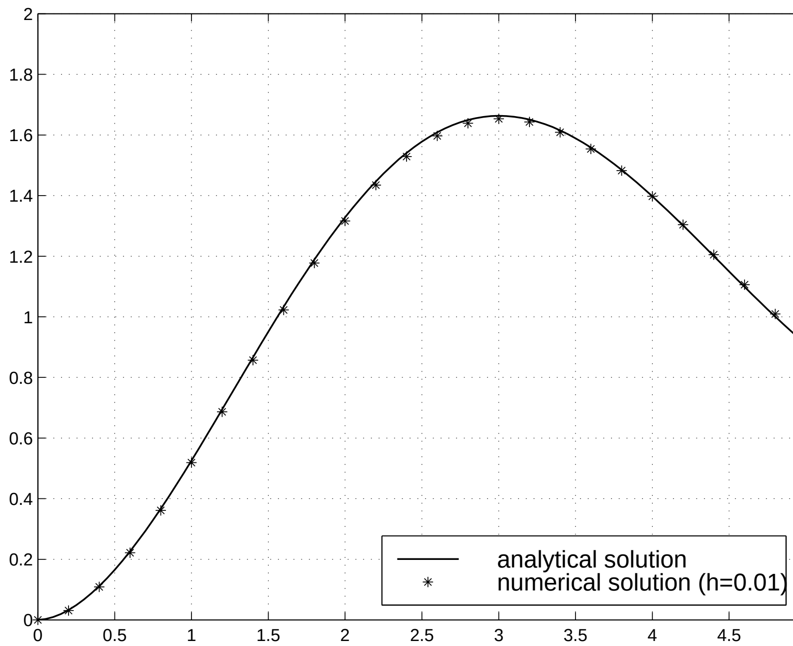

Figure 2: Solution of the problem $y^{(1.8)}(t) + y(t) = 1, y(0) = 0, y'(0) = 0$

which has the analytical solution

$$y(t) = t^{\alpha} E_{\alpha,\alpha+1}(-t^{\alpha}). \tag{120}$$

The numerical solution of the problem (118)–(119) can be found from the system (117), where we have $m = 2$, $\alpha_1 = \alpha$, $\alpha_2 = 0$, $n = 2$, $p_1 = p_2 = 1$, $B^{\alpha_1}_{N-n} = B^{\alpha}_{N-2}$, $B^{\alpha_2}_{N-n} = E_{N-2}$, $F_N = (\underbrace{1, 1, \ldots, 1}_{N})^T$. For these values, the system of algebraic equations for determining y_k, $k = 2, 3, \ldots, N$ takes on the form:

$$\left\{B^{\alpha}_{N-2} + E_{N-2}\right\}\left\{S_{0,1} Y_N\right\} = S_{0,1} F_N. \tag{121}$$

It should be also added that from the initial conditions we have $y_0 = y_1 = 0$.

The numerical solution of the problem (118)–(119) for $\alpha = 1.8$ is shown in **Fig. 2**.

8.3. Initial conditions in terms of integer-order derivatives

If the fractional derivatives in the equation (110), where $n - 1 < \alpha_m < n$, are Caputo derivatives, then the initial conditions are expressed in terms of classical integer-order derivatives and can be non-zero:

$$y^{(k)}(t_0) = c_k, \qquad k = 0, 1, \ldots, n-1. \tag{122}$$

The solution of the initial-value problem (110)–(122) can be written in the form

$$y(t) = y_*(t) + z(t), \tag{123}$$

where $y_*(t)$ is some known function, satisfying the conditions $y^{(k)}(t_0) = c_k$, $k = 0, 1, \dots, n-1$, and $z(t)$ is a new unknown function.

Substituting (123) into the equation (110) and the initial conditions (122), we obtain for the function $z(t)$ an initial-value problem with zero initial conditions, which can be solved as described in Section 8.2.

Example 3. Let us consider the following two-term fractional differential equation under non-zero initial conditions:

$$y^{(\alpha)}(t) + y(t) = 1, \tag{124}$$

$$y(0) = c_0, \quad y'(0) = c_1. \tag{125}$$

The analytical solution, obtained with the help of the Laplace transform of the Caputo fractional derivatives [12], is given by the expression

$$y(t) = c_0 E_{\alpha,1}(-t^\alpha) + c_1 t E_{\alpha,2}(-t^\alpha) + t^\alpha E_{\alpha,\alpha+1}(-t^\alpha). \tag{126}$$

To obtain numerical solution, we have first to transform the problem (124)–(125) to the problem with zero initial conditions. For this, let us introduce an auxiliary function $z(t)$, such that

$$y(t) = c_0 + c_1 t + z(t).$$

Substituting this expression into the equation (124) and in the initial conditions (125), we obtain the problem for finding $z(t)$:

$$z^{(\alpha)}(t) + z(t) = 1 - c_0 - c_1 t, \tag{127}$$

$$z(0) = 0, \quad z'(0) = 0. \tag{128}$$

The numerical solution of this problem can be found as described in Section 8.2, and the numerical solution $y(t)$ of the problem (124)–(125) is obtained using the relationship $y(t) = c_0 + c_1 t + z(t)$.

The numerical solution of the problem (124)–(125) for $\alpha = 1.8$, $c_0 = 1$, $c_1 = -1$ is shown in **Fig. 3**.

8.4. Initial conditions in terms of R-L fractional derivatives

Initial value problems for fractional differential equations with non-zero initial conditions in terms of Riemann-Liouville (R-L) derivatives, namely

$${}_aD_t^{\alpha-k-1} y(t)\Big|_{t\to a} = c_k, \qquad k = 0, 1, \dots, n-1, \tag{129}$$

can be also transformed to initial-value problems with zero initial condition. Such a transformation allows us to circumvent the difficulty consisting in the fact that there is still no known approximation for such initial conditions.

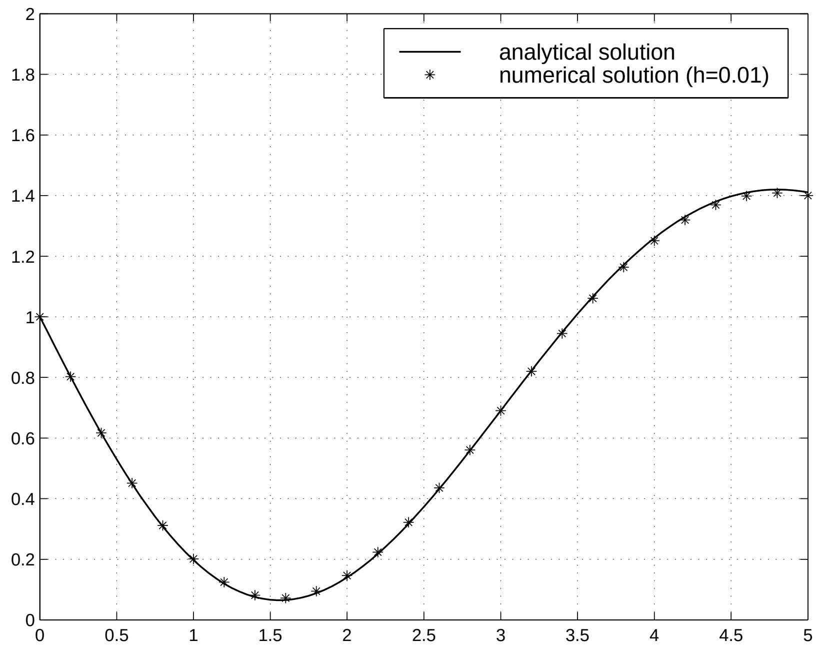

Figure 3: Solution of the problem $y^{(1.8)}(t) + y(t) = 1, y(0) = 1, y'(0) = -1$

Example 4. Let us consider the following two-term fractional differential equation under non-zero initial conditions:

$$y^{(\alpha)}(t) + y(t) = 1, \tag{130}$$

$$y^{(\alpha-1)}(0) = c_0, \quad y^{(\alpha-2)}(0) = c_1. \tag{131}$$

The analytical solution, obtained with the help of the Laplace transform of the Riemann–Liouville fractional derivative [12], is given by the expression

$$y(t) = c_0 t^{\alpha-1} E_{\alpha,\alpha}(-t^{\alpha}) + c_1 t^{\alpha-2} E_{\alpha,\alpha-1}(-t^{\alpha}) + t^{\alpha} E_{\alpha,\alpha+1}(-t^{\alpha}). \tag{132}$$

To obtain numerical solution, we have first to transform the problem (130)–(131) to the problem with zero initial conditions. For this, let us introduce an auxiliary function $z(t)$, such that

$$y(t) = c_0 t^{\alpha-1} + c_1 t^{\alpha-2} + z(t).$$

Substituting this expression into the equation (130) and into the initial conditions (131), we obtain the problem for finding $z(t)$:

$$z^{(\alpha)}(t) + z(t) = 1 - c_0 t^{\alpha-1} - c_1 t^{\alpha-2}, \tag{133}$$

$$z(0) = 0, \quad z'(0) = 0. \tag{134}$$

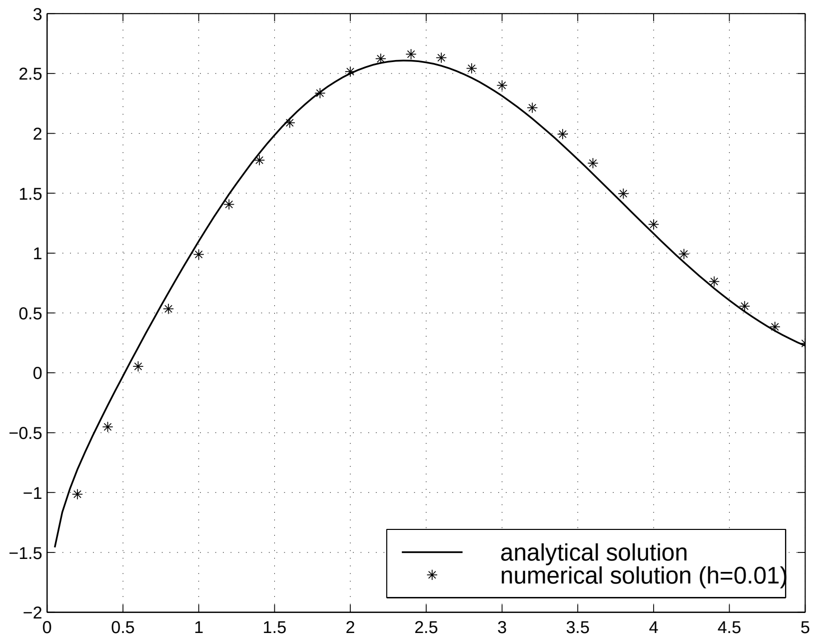

Figure 4: Solution of the problem $y^{(1.8)}(t)+y(t)=1; y^{(0.8)}(0)=1; y^{(-0.2)}(0)=-1$.

The numerical solution of this problem can be found as described in Section 8.2, and the numerical solution $y(t)$ of the problem (130)–(131) is obtained using the relationship $y(t)=c_0t^{\alpha-1}+c_1t^{\alpha-2}+z(t)$.

The numerical solution of the problem (130)–(131) for $\alpha=1.8$, $c_0=1$, $c_1=-1$ is shown in **Fig. 4**.

8.5. Nonlinear FDEs

The triangular strip matrices can be useful also for solving fractional differential equations of a general form. Let us write, for example, an equation with left-sided fractional derivatives $y^{(\alpha_i)}(t)={}_aD_t^{\alpha_i}y(t)$:

$$y^{(\alpha_1)}(t)=f(t,\,y^{(\alpha_2)}(t),\,y^{(\alpha_3)}(t),\,\ldots\,,y^{(\alpha_k)}(t)), \tag{135}$$

$$(0<\alpha_1<\alpha_2<\ldots<\alpha_k\le n.)$$

assuming that the initial conditions are already transformed to zero initial conditions.

Replacing all fractional derivatives in the equation (135) with their discrete analogues and utilizing zero initial conditions, we obtain a nonlinear algebraic system

$$B_N^{\alpha_1}Y_N=f(Et_N,\,B_N^{\alpha_2}Y_N,\,B_N^{\alpha_3}Y_N,\,\ldots\,,B_N^{\alpha_k}Y_N), \tag{136}$$

$$y_j=0,\quad j=1,\,2,\,\ldots\,,n-1,$$

where $Y_N = (y_0, y_1, \ldots, y_N)^T$, $t_N = (t_0, t_1, \ldots, t_N)^T$, $y_j = y(t_j)$, $t_j = jh$, $(j = 0, 1, \ldots, N)$, and E is $(N+1) \times (N+1)$ unit matrix.

9. Conclusion

The suggested approach, using the triangular strip matrices, provides:

- a uniform approach to discretization of derivatives of arbitrary real order, including the classical integer-order derivatives, and various types of fractional derivatives, including the left- and right-sided derivatives, and the sequential fractional derivatives;
- a uniform approach to numerical solution of differential equations of integer order and of fractional order;
- a convenient language for discretization of differential equations of arbitrary real order;
- a method for numerical solution of initial value problems and boundary value problems for ordinary differential equations of arbitrary real order;
- a possible method for numerical solution of non-linear differential equations of arbitrary real order.

The triangular matrix approach can also be used for obtaining new quadrature formulas for fractional integrals. For this, any approximation of fractional derivatives should be written in the form of a triangular strip matrix, inversion of which gives the corresponding quadrature formula for fractional integrals.

Similarly, new approximations of fractional derivatives can be obtained by inverting the triangular strip matrices, corresponding to quadrature formulas for fractional integrals.

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