Numerical methods of the fractional calculus

(continued II)

Computation of binomial coefficients

The following coefficients appear frequently (e.g., in G1):

$$w_k^{(\alpha)} = (-1)^k \binom{\alpha}{k}, \qquad k = 0, 1, 2, \ldots,$$

It is possible to use the following recurrence:

$$w_0^{(\alpha)} = 1; \quad w_k^{(\alpha)} = \left(1 - \frac{\alpha+1}{k}\right) w_{k-1}^{(\alpha)}, \quad k = 1, 2, 3, \ldots$$

Possible MATLAB code:

```
function y=bcrecur(a, n)
y=cumprod([1, 1 - ((a+1) ./ [1:n])]);
```


Computation of binomial coefficients

```
>> bcrecur(1.8,10)
ans =
  Columns 1 through 8
    1.0000   -1.8000    0.7200    0.0480    0.0144    0.0063    0.0034    0.0020
  Columns 9 through 11
    0.0013    0.0009    0.0007
>> bcrecur(2,10)
ans =
     1    -2     1     0     0     0     0     0     0     0     0
>> bcrecur(3,10)
ans =
  Columns 1 through 8
    1.0000   -3.0000    3.0000   -1.0000         0         0         0         0
  Columns 9 through 11
         0         0         0
```

for each routine make a qualitative check

and check the backward compatibility for integer orders

Computation of the Mittag-Leffler function

(1) Directly using the definition:

```
function y=mitlef(alpha,beta,z,N)

%   Evaluation of the Mittag-Leffler function in two parameters:
%   E_{alpha,beta}(z), using its series expansion.
%
% PARAMETERS:
%   alpha   - first index (scalar), beta   - second index (scalar)
%   z    - row vector of values of the function argument
%   N    - number of terms in the power expansion (scalar)
%          OPTIONAL, default N=100.

if nargin<4, N=100; end

m1=max(size(z));
m2=min(size(z));
if m2>1, z=z(1,:); end

k=repmat((1:N)',1,m1);
t=repmat(z,N,1);
t=t.^(k-1);
a=repmat(gamma(alpha*((1:N)'-1)+beta), 1, m1);
y=sum(t./a);
```


Computation of the Mittag-Leffler function

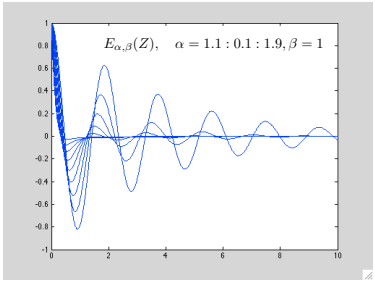
(2) There is a routine at MATLAB File Exchange:

http://www.mathworks.com/matlabcentral/fileexchange/loadFile.do?objectId=8738&objectType=FILE

`MLF(alpha,beta,Z,P)`
is the Mittag-Leffler function $E_{alpha,beta}(Z)$ evaluated with accuracy 10^{-P} for each element of `Z`.

Computation of the Mittag-Leffler function

(2) There is a routine at MATLAB File Exchange:



Computation of fractional derivatives

Hany Farid:

```
%%% FRACTIONAL DERIVATIVES (12.14.00)
%%% Hany Farid (farid@cs.dartmouth.edu | www.cs.dartmouth.edu/~farid)
%%%

clear;
set( gcf, 'Renderer', 'zbuffer' );

dim    = 256;
ramp   = [-dim:dim-1];
ramp   = pi * ramp/dim;
f      = exp( -(ramp.^2)/(0.5) );          % GAUSSIAN
f      = f - mean(f);                      % ZERO-MEAN:AVOID DISCONTINUITY
F      = fftshift( fft( f ) );             % FOURIER TRANSFORM

%%% CYCLE THROUGH FRACTIONAL DERIVATIVES
for n = 0 : 0.1 : 4
        Fn     = (j * ramp).^n .* F;       % MULIPLY BY RAMP^n
        fn     = ifft( ifftshift(Fn) );    % INVERSE FOURIER TRANSFORM
        if( mean( abs(imag(fn)) ) > 1e-5 )  % FOR REAL SIGNALS
          fprintf( 1, 'Non-zero imaginary component (%f)\n',mean(abs(imag(fn))) );
          return;
        end
        fn     = fn / max(abs(fn));        % NORMALIZE
        t = sprintf( '%0.2f', n );         % PLOT...
        plot( real(fn) );
        text( 20, 1.0, sprintf( '%0.2f', n ) );
        axis( [0 2*dim -1.2 1.2] ); axis square;
        h = line( [dim dim], [-1.2 1.2] );
        set( h, 'LineStyle', ':' );
        drawnow;
end
```


Digital fractional differentiators

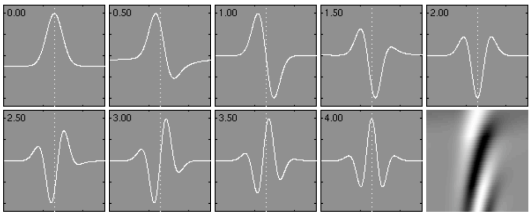
Ivo Petras -- IIR version:

```
function sysdfod=dfod1(n,T,a,r)
% sysdfod=dfod1(n,T,a,r): new digital fractional order differentiator
%                         and integrator
%
% Output: =>
% Discrete system in the form of the rational approximation - IIR filter
% obtained by continued fraction expansion of a new generating function.
%
% Inputs: <=
% n: order of truncation n:=[1 2 3 4 5]
% T: sampling period in [sec]
% a: weighting factor between Euler and Tustin rules ( 0 <= a <= 1 )
%    0 - Euler rule, 1 - Tustin rule, 1/2 - Al-Alaoui rule, etc.
% r: approximated fractional order (s^r), r is generally real number
%
% Author: Dr. Ivo Petras (ivo.petras@mail.com)
% URL: http://ivopetras.tripod.com/
%
% Note: This approach is based on the original Al-Alaoui's work.
%
% Copyright (c), 2003.
%
k = ((1+a)/T)^r;
if (n>5)
    n=5;
    disp('Maximal term is clipped to 5!');
end
if (n==1)
    b0=2/(a*r+a*r-1);
    b1=-(r+a*r+a+1)/(a*r+a*r-1);
    a0=2/(a*r+a*r-1);
    a1=1;
    sys1=tf(k*[b0 b1],[a0 a1],T,'Variable', 'z^-1');
    sysdfod=sys1;
elseif (n==2)
    b0=12/(a^2*r^2+3*a^2*r+2*a*r^2+2*a*r^2-2*a-3*r+2);
    b1=(12*a-6*r-6*a*r-12)/(a^2*r^2+3*a^2*r+2*a*r^2+2*a*r^2-2*a-3*r+2);
    b2=(a^2*r^2-3*a^2*r+2*a*r^2+2*a*r^2-2*a+3*r+2*a^2+2*a^2*r^2+3*a^2*r+2*a*r^2+2*a*r^2-2*a-3*r+2);
    a0=12/(a^2*r^2+3*a^2*r+2*a*r^2+2*a*r^2-2*a-3*r+2);
    a1=(12*a+6*r+6*a*r-12)/(a^2*r^2+3*a^2*r+2*a*r^2+2*a*r^2-2*a-3*r+2)
    a2=1;
    sys2=tf(k*[b0 b1 b2],[a0 a1 a2],T,'Variable', 'z^-1');
    sysdfod=sys2;
elseif (n==3)
    b0=120/(a^3*r^3+6*a^3*r^2+3*a^2*r^3+11*a^3*r+6*a^2*r^2+3*a*r^3+3*a^3+6*a^3-27*a^2*r-6*a*r^2+r^3-54*a^2-27*a*r-6*r^2+54*a+11*r-6);
```


Computation of fractional derivatives

Hany Farid (www.cs.dartmouth.edu/~farid):

Zeroth through fourth-order normalized derivatives of a Gaussian in steps of 0.5.



Digital fractional differentiators

YangQuan Chen -- IIR version:

```
% New DFOD
function sysdfodnewdfod(n,T,a,s)
% n order of truncation (1 2 3 ...)
% T sampling period in sec.
% a weighting factor between Simpson and Tustin ( 0 <= a <= 1 )
z2=(3*a-2*sqrt(3*a))/(3-a);b=sqrt(3*(1-a))*z;
if (n==1)
    b1=-(2*b/2*s+b*2-1);
    b0=2*b/(2*b/2*s+b*2-1);
    a0=1;
    a1=1;
    a0=2*b/(2*b/2*s+b*2-1);
    sys1=tf([b0 b1],[a0 a1],T,'Variable', 'z^-1');
    sysdfod=sys1;
elseif (n==2)
    den2=(8*b^4*s^4+12*b^4*s^3+24*b^4*s^2-3*b^4*s-12*b^4*s^3-b^6-36*b^4*s^2+3*b^4+3*b^4+18*b^2*s^2-9*b^2*s+9*b^2+9*s-9);
    b2=(18*b^2*s^2+3*b^4*s^2+6*b^2*s^2+3*b^4*s+2*b^6*s^2-12*b^4*s^3-3*b^4+12*b^4*s^3-3*b^4*s+9*b^2*s-9*s+9*b^4+9*b^2-9*s)/den2;
    b1=(-24*s^2*b^3-24*s^3*b^5-12*b^3*s^3+18*b+36*s*b^3+18*s^6-6*s*b^5+24*s^2*b^5-6*b^5)/den2;
    b0=(18+24*b^4*s^2-6*b^4-36*b^2)/den2;
    a
    a2=1;
    a1=(-24*s^2*b^3+24*s^3*b^5-12*b^3*s^3+18*b-36*s*b^3+18*s^6-6*s*b^5+24*s^2*b^5-6*b^5)/den2;
    a0=(18+24*b^4*s^2-6*b^4-36*b^2)/den2;
    sys2=tf([b0 b1 b2],[a0 a1 a2],T,'Variable', 'z^-1');
    sysdfod=sys2;
elseif (n==3)
den3=(64*b^12*s^7+192*b^12*s^6+144*b^12*s^5-48*b^12*s^4-192*b^10*s^6-64
  -1440*b^10*s^5-1296*b^12*s^2-162*b^10*s^4+11*b^12*s+720*b^10*s^3+720
  -90*b^10*s+2520*b^8*s^3-1080*b^10-3240*b^8*s^2-480*b^6*s^6-675*b^8*s
  -1440*b^6*s^5+1440*b^8*s+1080*b^6*s^4+1440*b^4*s^6+1350*b^6*s^3-1350
b3=-(675*s^2*b^8+3*b^8*s^4-240*b^6*s^4+540*b^6*s+108*b^10*s^3+360*b^2*s+2700*s
  -8100*b^6*s^2-2475*b^4*s-720*b^4*s^3+9480*b^4*s^2+180*b^6*s^4+480*b^6
  -324*b^8*s^2-675*b^8*s-720*b^4*s^4+675*s^6+90*s^4+180*b^4*s^2-720*
  1440*b^10*s^5+192*b^10*s^6+11*b^12*s+12*b^12*s^2-864*b^12*s^3+48*b^12
  +64*b^12*s^7-36*b^12/den3;
b2=-(-5400*b+6300*b^3-1440*b^5-3240*b^7-2700*s*b-360*b^9+2700*s*b^3-1800
  -1800*s*b^7-1224*b^7*s^2+720*s^3*b^3-288*s^4*b^7+1280*s*b-3280
  -480*b^11*s^3+960*b^11*s^5-384*b^11*s^6+264*b^11*s^2-384*b^11*s^4-36
b1= (-6300*b^4-8460*b^6+3*s1080*b^4*s-8100*b^2*s+3420*b^6+360*b^4*s^2-
  -1440*b^8*s^4-1440*b^10*s^4+960*b^10*s^5-13680*b^6*s^2+6750*b^2-480
  +1350-720*b^10*s^2)/den3;
b0= (-10800*b^3+10800*b^5+8100*b-2160*b^7-960*b^9*s^4+480*b^9*s^2-60*b
% denominator
a3=1;
a2=(5400*b-6300*b^3+1440*b^5-3240*b^7-2700*s*b+360*b^9+2700*s*b^3-1800
  -1800*s*b^7+1224*b^7*s^2+720*s^3*b^3-288*s^4*b^7+1280*s*b^3-3280
  -480*b^11*s^3+960*b^11*s^5-384*b^11*s^6+264*b^11*s^2-384*b^11*s^4-36
a1=(6300*b^4-8460*b^6+3*s1080*b^4*s-8100*b^2*s-3420*b^6-360*b^4*s^2-
  -1440*b^8*s^4+1440*b^10*s^4+960*b^10*s^5+13680*b^6*s^2-6750*b^2-480*b^10*s^3+2430*b^8-8360*b^8*s^2+480*b^10*s^2
  +1350-720*b^10*s^2)/den3;
```


Digital fractional differentiators

Ivo Petras -- Finite Impulse Response (FIR) version:

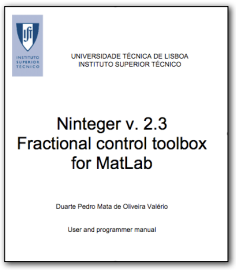
```
function sysdfod=dfod2(n,T,r)
% sysdfod=dfod2(n,T,r): digital fractional order differentiator
%                       and integrator
%
% Output: =>
% Discrete system in the form of the FIR filter of the order n
% obtained by power series expansion of the backward difference.
%
% Inputs: <=
% n: order of truncation (min n=100 is recommended)
% T: sampling period in [sec]
% r: approximated fractional order (s^r), r is generally real number
%
% Author: Dr. Ivo Petras (ivo.petras@mail.com)
% URL: http://ivopetras.tripod.com/
%
% Note: differentiator  -> nonrecusrive approximation
%       integrator      -> recursive approximation
%
% Copyright (c), 2003.
%
if r>0
   bo=cumprod([1,1-((r+1)./[1:n])]);
   sysdfod=filt(bo,[T^r],T);
end
if r<0
   bo=cumprod([1,1-((-r+1)./[1:n])]);
   sysdfod=filt([T^-r],bo,T);
end
%
```


Digital fractional differentiators

YangQuan Chen --

Oustaloup's recursive approximation for fractional order differentiators:

NINTEGER Toolbox by Duarte Valerio



NINTEGER Toolbox by Duarte Valerio

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NINTEGER Toolbox by Duarte Valerio

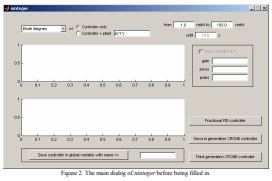


Figure 2. The main dialog of ninteger before being filled in.

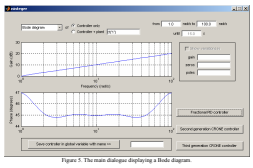


Figure 5. The main dialogue displaying a Bode diagram.

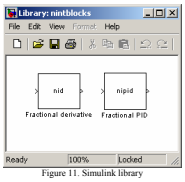


Figure 11. Simulink library

Sample applications

Fractional derivatives of order 1/2 can appear even within the context of the integer-order models!

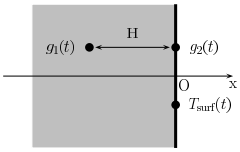
Heat flux in a blast furnace wall

$$\hat{c}\hat{\rho}\frac{\partial T}{\partial t} = \hat{\lambda}\frac{\partial^2 T}{\partial x^2}, \qquad (t>0, \quad -\infty < x < 0)$$

$$T(0,x) = T_0$$

$$T(t,0) = T_{\text{surf}}(t)$$

$$\left|\lim_{x\to-\infty} T(t,x)\right| < \infty$$



t - is time [s],
x - is the spatial coordinate in the direction of heat conduction [m],
$\hat{c}$ - is heat capacity $[J\,kg^{-1}\,K^{-1}]$,
$\hat{\rho}$ - is mass density $[kg\,m^{-3}]$,
$T(t,x)$ - is temperature $[K]$,
$\hat{\lambda}$ - is the coefficient of heat conductivity $[W\,m^{-1}\,K^{-1}]$

Heat flux in a blast furnace wall

Introduce an auxiliary function

$$u(t,x) = T(t,x) - T_0.$$

which must be a solution of the following problem:

$$\hat{c}\hat{\rho}\frac{\partial u}{\partial t} = \hat{\lambda}\frac{\partial^2 u}{\partial x^2}, \qquad (t>0, \quad -\infty < x < 0)$$

$$u(0,x) = 0$$

$$u(t,0) = T_{\text{surf}}(t) - T_0$$

$$\left|\lim_{x\to-\infty} u(t,x)\right| < \infty$$

Heat flux in a blast furnace wall

The Laplace transform gives:

$$s\,\hat{c}\hat{\rho}U(s,x) = \hat{\lambda}\frac{d^2U(s,x)}{dx^2}$$

The solution bounded at $x \to -\infty$ is:

$$U(s,x) = U(s,0)\exp\left(x\sqrt{\frac{\hat{c}\hat{\rho}s}{\hat{\lambda}}}\right)$$

Then we have

$$\frac{dU}{dx}(s,x) = U(s,0)\sqrt{\frac{\hat{c}\hat{\rho}s}{\hat{\lambda}}}\,\exp\left(x\sqrt{\frac{\hat{c}\hat{\rho}s}{\hat{\lambda}}}\right)$$

Heat flux in a blast furnace wall

$$U(s,x) = U(s,0)\exp\left(x\sqrt{\frac{\hat{c}\hat{\rho}s}{\hat{\lambda}}}\right) \qquad \frac{dU}{dx}(s,x) = U(s,0)\sqrt{\frac{\hat{c}\hat{\rho}s}{\hat{\lambda}}}\,\exp\left(x\sqrt{\frac{\hat{c}\hat{\rho}s}{\hat{\lambda}}}\right)$$

$$\frac{1}{\sqrt{s}}\frac{dU}{dx}(s,0) = \sqrt{\frac{\hat{c}\hat{\rho}}{\hat{\lambda}}}\,U(s,0)$$

The inverse Laplace transform gives:

$${}_0D_t^{-1/2}\frac{\partial u}{\partial x}(t,0) = \sqrt{\frac{\hat{c}\hat{\rho}}{\hat{\lambda}}}\,u(t,0)$$

After fractional integration of both sides:

$$\frac{\partial u}{\partial x}(t,0) = \sqrt{\frac{\hat{c}\hat{\rho}}{\hat{\lambda}}}\,{}_0D_t^{1/2}u(t,0)$$

Heat flux in a blast furnace wall

$$\frac{\partial u}{\partial x}(t,0) = \sqrt{\frac{\hat{c}\hat{\rho}}{\hat{\lambda}}}\,{}_0D_t^{1/2}u(t,0) \qquad u(t,x) = T(t,x) - T_0,$$

$$q_A(t) = \sqrt{\hat{c}\hat{\rho}\hat{\lambda}}\,{}_0D_t^{1/2}g(t), \quad g(t) = T_{\text{surf}}(t) - T_0$$

$$q_A(t) = \hat{\lambda}\frac{\partial T}{\partial x}(t,0)$$

the heat flux

Heat flux in a blast furnace wall

Numerical solution: short memory principle

$$q_A(t) \approx \tilde{q}_A(t) = \sqrt{\hat{c}\hat{\rho}\hat{\lambda}}\,{}_{(t-L)}D_t^{1/2}g(t)$$

Normalized error:

$$\delta_0 = \frac{|q_A(t) - \tilde{q}_A(t)|}{M} = \frac{1}{\sqrt{L}\,\Gamma(\frac{1}{2})}, \quad M = \max_{[0,\infty]}|g(t)|$$

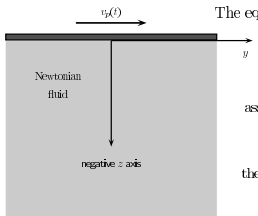
Condition on the memory length: $L \geq \dfrac{1}{\pi\delta_0^2}$

Calculate using:

$${}_{(t-L)}D_t^{1/2}g(t) = \tau^{-\alpha}\sum_{i=0}^{N(t)} c_i g(t - i\tau),$$

$$N(t) = \min\left\{\left[\frac{t}{\tau}\right], \left[\frac{L}{\tau}\right]\right\}, \quad c_i = (-1)^i\binom{1/2}{i}$$

A plate on a Newtonian fluid



The equation of motion of the fluid is

$$\rho\frac{\partial v}{\partial t} = \mu\frac{\partial^2 v}{\partial z^2}, \qquad (0 < t < \infty, \quad -\infty < z < 0)$$

assume that initially the fluid is in equilibrium state.

$$v(0,z) = 0, \qquad (-\infty < z < 0)$$

the influence of the plate's motion vanishes for $z \to \infty$:

$$v(t,-\infty) = 0, \qquad (0 < t < \infty)$$

The fluid's velocity at $z = 0$ is equal to the given velocity of the plate:

$$v(t,0) = v_p(t)$$

A plate on a Newtonian fluid

Applying the Laplace transform gives:

$$\rho s V(s,z) = \mu\frac{d^2V(s,z)}{dz^2}$$
$$V(s,0) = V_p(s),$$
$$V(s,-\infty) = 0,$$

The solution of this problem is:

$$V(s,z) = V_p(s)\exp\left(z\sqrt{\frac{\rho s}{\mu}}\right)$$

A plate on a Newtonian fluid

By differentiation we obtain:

$$\frac{dV(s,z)}{dz} = \sqrt{\frac{\rho s}{\mu}} V_p(s) \exp\left(z\sqrt{\frac{\rho s}{\mu}}\right) = \sqrt{\frac{\rho s}{\mu}} V(s,z)$$

Knowing the velocity, we can obtain the shear stress:

$$\sigma(t,z) = \mu \frac{\partial v(t,z)}{\partial z}$$

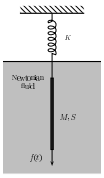
which in terms of the Laplace transforms reads:

$$\overline{\sigma}(s,z) = \mu \frac{dV(s,z)}{dz} = \sqrt{\mu\rho s}\, V(s,z).$$

and therefore, in time domain we have:

$$\sigma(t,z) = \sqrt{\mu\rho}\, {}_0D_t^{1/2} v(s,z)$$

Motion of an immersed plate



Considering the forces we have:

$$My''(t) = f(t) - Ky(t) - 2S\sigma(t,0)$$

Using the relationships

$$\sigma(t,z) = \sqrt{\mu\rho}\, {}_0D_t^{1/2} v(s,z) \qquad v_p(t,0) = y'(t)$$

we arrive at the following equation:

$$Ay''(t) + B\, {}_0D_t^{3/2} y(t) + Cy(t) = f(t) \qquad (t>0),$$

$$A = M, \quad B = 2S\sqrt{\mu\rho}, \quad C = K,$$

add initial conditions! $\quad y(0) = 0, \qquad y'(0) = 0$

Motion of an immersed plate

Sample numerical solution

```
% Script for solving Bagley-Torvik equation with zero
initial conditions:

clear all

alpha=1.5;

% Numerical solution:
h=0.075;
T=0:h:30;
N=30/h + 1;
B=zeros(N,N);

% (1) Make the main matrix B1_{N}^{alpha}:
bc=fliplr(bcrecur(alpha,N-1));
for k=1:N
   B1(k,1:k)=bc((N-k+1):N);
end
```


Motion of an immersed plate

Sample numerical solution

```
% (2) Make the main matrix B2_{N}^{2}:
alpha=2;
bc=fliplr(bcrecur(alpha,N-1));
for k=1:N
   B2(k,1:k)=bc((N-k+1):N);
end

% (3) Make the matrix for the entire equation:
B=h^(-2)*B2 + h^(-1.5)*B1 + eye(size(B1));

% Make right-hand side:
F=8*(T<=1)';
```


Motion of an immersed plate

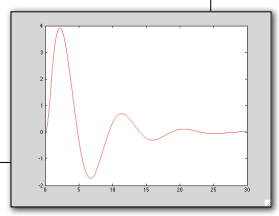
Sample numerical solution

```
% Make right-hand side:
F=8*(T<=1)';

% (3) Use zero initial conditions:
F(1)=0;
F(2)=0;
B(1,1)=1;
B(2,1)=-1/h; B(2,2)=1/h;

% (4) Solve the system BY=F:
Y=B\F;

% Plot the solution:
figure(1)
plot(T,Y,'r')
```



Short memory principle for FDEs

Consider the problem:

$${}_0D_t^{3/2} y(t) + y(t) = f(t), \qquad (t>0)$$

$$y(0) = y'(0) = 0$$

for the following RHS:

1. $f(t) \equiv 1$
2. $f(t) = te^{-t}$
3. $f(t) = t^{-1}e^{-1/t}$
4. $f(t) = e^{-t}\sin(0.2t)$

Short memory principle for FDEs

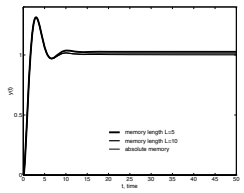


Figure 8.8: *Solution of the problem (8.56) for* $f(t) \equiv 1$

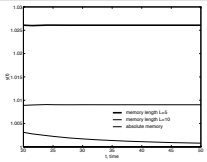


Figure 8.9: *Solution of the problem (8.56) for* $f(t) \equiv 1$ *(zoom)*

Short memory principle for FDEs

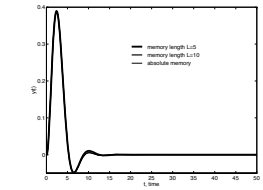


Figure 8.10: *Solution of the problem (8.56) for* $f(t) = te^{-t}$

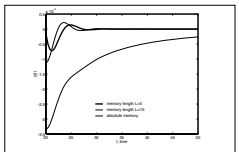


Figure 8.11: *Solution of the problem (8.56) for* $f(t) = te^{-t}$ *(zoom)*

Short memory principle for FDEs

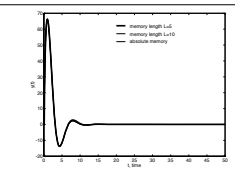


Figure 8.12: *Solution of the problem (8.56) for* $f(t) = t^{-1/2}e^{-1/t}$

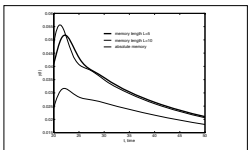


Figure 8.13: *Solution of the problem (8.56) for* $f(t) = t^{-1/2}e^{-1/t}$ *(zoom)*

Short memory principle for FDEs

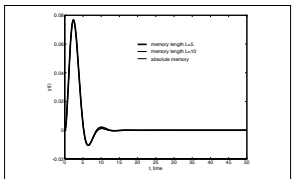


Figure 8.14: *Solution of the problem (8.56) for* $f(t) = e^{-t}\sin(0.2t)$

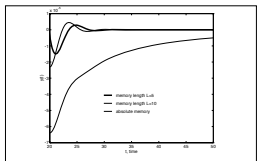


Figure 8.15: *Solution of (8.56) for* $f(t) = e^{-t}\sin(0.2t)$ *(zoom)*