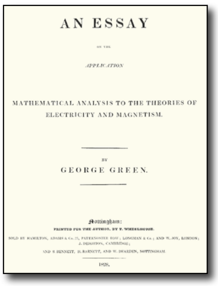
Green's functions



George Green
(14 July 1793 – 31 May 1841)

In 1828 Green privately published "An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism". The essay introduced several important concepts, among them a theorem similar to modern Green's theorem, the idea of potential functions as currently used in physics, and the concept of what are now called Green's functions. It gained him admittance to Cambridge as an undergraduate in 1833. He graduated in 1837 and was elected to a fellowship in 1839, two years before his death. The fellowship was for bachelors; Green qualified because he had never formally married the mother of his six children. No portrait of Green was ever made...

Green's function
Classical case (recall)

Linear differential equation of integer order n:

$$\sum_{k=0}^{n} p_k(t) y^{(k)}(t) = f(t), \quad t \in [a,b]$$

Structure of solution:

$$y(t) = y_H(t) + y_f(t)$$

$$y_f(t) = \int_a^b G(t,\tau) f(\tau) d\tau$$

Green's function
Classical case (recall)

Definition of $G(t,\tau)$:

1. Partial derivatives $G_t^{(k)}(t,\tau), \quad (k = 0, \ldots, n)$ exist and are continuous with respect to both variables in triangles $a \le t \le \tau \le b$ and $a \le \tau \le t \le b$.

2. As function of t, $G(t,\tau)$ satisfies equation
$$\sum_{k=0}^{n} p_k(t) y^{(k)}(t) = 0$$

Green's function
Classical case (recall)

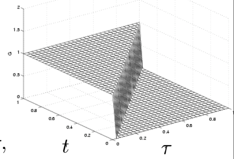
3. $G(t,\tau)$ and its derivatives $G_t^{(k)}(t,\tau), (k = 0, \ldots, n-2)$ are continuous in the square $a \le t, \tau \le t$ with respect to both variables,

4. For $a < \tau < t$
$$G_t^{(n-1)}(\tau+0,\tau) - G_t^{(n-1)}(\tau-0,\tau) = \frac{1}{p_n(\tau)}$$
[jump of (n-1)-th derivative]

Green's function
Classical case (recall)

Consider the simplest example:

$$\begin{cases} y'(t) = f(t), & t \in [0,T] \\ y(0) = 0 \end{cases}$$



The solution is:

$$y(t) = \int_0^t f(\tau) d\tau = \int_0^T G(t,\tau) f(\tau) d\tau,$$

$$G(t,\tau) = \begin{cases} 1, & 0 \le \tau \le t \\ 0 & t < \tau \le T \end{cases} = H(t-\tau)$$

Heaviside function

Green's function
Classical case (recall)

The LT can be used to obtain Green's function:

$$\begin{cases} y'(t) = f(t), & t \in [0,T] \\ y(0) = 0 \end{cases}$$

Consider the same equation with delta function in RHS:

$$G'(t) = \delta(t)$$

$$s\,g(s) = 1 \;\rightarrow\; g(s) = \frac{1}{s} \;\rightarrow\; G(t) = H(t) = \begin{cases} 1, & t \ge 0 \\ 0, & t < 0 \end{cases}$$

Heaviside

$$y(t) = \int_0^T G(t-\tau) f(\tau) d\tau$$

Fractional Green's function

Initial value problem for a linear FODE

$$\boxed{{}_0\mathcal{L}_t y(t) = f(t); \qquad \left[{}_0\mathcal{D}_t^{\sigma_k - 1} y(t)\right]_{t=0} = 0, \quad (k = 1, \ldots, n)}$$

where

$$_a\mathcal{L}_t y(t) \equiv {}_a\mathcal{D}_t^{\sigma_n} y(t) + \sum_{k=1}^{n-1} p_k(t)\, {}_a\mathcal{D}_t^{\sigma_{n-k}} y(t) + p_n(t) y(t),$$

$$_a\mathcal{D}_t^{\sigma_k} \equiv {}_aD_t^{\alpha_k}\, {}_aD_t^{\alpha_{k-1}} \ldots {}_aD_t^{\alpha_1}; \quad {}_a\mathcal{D}_t^{\sigma_k - 1} \equiv {}_aD_t^{\alpha_k - 1}\, {}_aD_t^{\alpha_{k-1}} \ldots {}_aD_t^{\alpha_1};$$

$$\sigma_k = \sum_{j=1}^{k} \alpha_j, \quad (k = 1, 2, \ldots, n); \quad 0 \le \alpha_j \le 1, \quad (j = 1, 2, \ldots, n).$$

Fractional Green's function

Definition

Function $G(t,\tau)$ satisfying the following conditions

a) ${}_\tau\mathcal{L}_t G(t,\tau) = 0$ for every $\tau \in (0,t)$;

b) $\lim_{\tau \to t-0}({}_\tau\mathcal{D}_t^{\sigma_k - 1} G(t,\tau)) = \delta_{k,n}, \quad k = 0, 1, \ldots, n,$

($\delta_{k,n}$ is Kronecker's delta);

c) $\lim\limits_{\substack{\tau, t \to +0 \\ \tau < t}} ({}_\tau\mathcal{D}_t^{\sigma_k} G(t,\tau)) = 0, \quad k = 0, 1, \ldots, n-1$

is called Green's function of equation ${}_0\mathcal{L}_t y(t) = f(t)$

Leopold Kronecker
(1823-1891)

Kronecker believed that mathematics should deal only with finite numbers and with a finite number of operations.

Fractional Green's function

Main property

Solution of the problem

$${}_0\mathcal{L}_t y(t) = f(t); \qquad \left[{}_0\mathcal{D}_t^{\sigma_k - 1} y(t)\right]_{t=0} = 0, \quad (k = 1, \ldots, n)$$

is given by

$$y(t) = \int_0^t G(t,\tau) f(\tau) d\tau$$

Fractional Green's function

Main property (1)

Indeed, consider ${}_0\mathcal{D}_t^{\sigma_1} y(t), \quad {}_0\mathcal{D}_t^{\sigma_2} y(t), \quad \ldots, \quad {}_0\mathcal{D}_t^{\sigma_n} y(t)$

$$\begin{aligned}
{}_0\mathcal{D}_t^{\sigma_1} y(t) &= {}_0D_t^{\alpha_1} \int_0^t G(t,\tau) f(\tau) d\tau \\
&= \int_0^t {}_\tau D_t^{\alpha_1} G(t,\tau) f(\tau) d\tau + \lim_{\tau \to t-0} {}_\tau D_t^{\alpha_1 - 1} G(t,\tau) f(\tau) \\
&= \int_0^t {}_\tau\mathcal{D}_t^{\sigma_1} G(t,\tau) f(\tau) d\tau
\end{aligned}$$

Fractional Green's function

Main property (2)

$$\begin{aligned}
{}_0\mathcal{D}_t^{\sigma_2} y(t) &= {}_0D_t^{\alpha_2}({}_0D_t^{\alpha_1} y(t)) = {}_0D_t^{\alpha_2} \int_0^t {}_\tau D_t^{\alpha_1} G(t,\tau) f(\tau) d\tau \\
&= \int_0^t {}_\tau D_t^{\alpha_2}({}_\tau D_t^{\alpha_1} G(t,\tau)) f(\tau) d\tau \\
&\quad + \lim_{\tau \to t-0} {}_\tau D_t^{\alpha_2 - 1}({}_\tau D_t^{\alpha_1} G(t,\tau)) f(\tau) \\
&= \int_0^t {}_\tau\mathcal{D}_t^{\sigma_2} G(t,\tau) f(\tau) d\tau
\end{aligned}$$

and similarly up to ${}_0\mathcal{D}_t^{\sigma_{n-1}} y(t) = \int_0^t {}_\tau\mathcal{D}_t^{\sigma_{n-1}} G(t,\tau) f(\tau) d\tau$

Fractional Green's function

Main property (n)

But:

$$\begin{aligned}
{}_0\mathcal{D}_t^{\sigma_n} y(t) &= {}_0D_t^{\alpha_n}({}_0\mathcal{D}_t^{\sigma_{n-1}} y(t)) = {}_0D_t^{\alpha_n} \int_0^t {}_\tau\mathcal{D}_t^{\sigma_{n-1}} G(t,\tau) f(\tau) d\tau \\
&= \int_0^t {}_\tau D_t^{\alpha_n}({}_\tau\mathcal{D}_t^{\sigma_{n-1}} G(t,\tau)) f(\tau) d\tau + \lim_{\tau \to t-0} {}_\tau D_t^{\alpha_n - 1}({}_\tau\mathcal{D}_t^{\sigma_{n-1}} G(t,\tau)) f(\tau) \\
&= \int_0^t {}_\tau\mathcal{D}_t^{\sigma_n} G(t,\tau) f(\tau) d\tau + f(t)
\end{aligned}$$

Therefore,

$${}_0\mathcal{L}_t y(t) = \int_0^t {}_\tau\mathcal{L}_t G(t,\tau) f(\tau) d\tau + f(t) = f(t).$$

Fractional Green's function
Constructing solutions of equations with zero RHS

Motivation: consider an integer order example:

$$G''(t) + a^2 G(t) = \delta(t)$$

$$g(s)(s^2 + a^2) = 1 \qquad g(s) = \frac{1}{s^2 + a^2}$$

$$G(t) = \frac{1}{a}\sin at$$

$G^{(1)}(t)$ $\qquad$ $G^{(0)}(t)$

$$\psi_1(t) = \cos at \qquad \psi_2(t) = \frac{1}{a}\sin at$$

$$\psi_1(0) = 1 \qquad \psi_2(0) = 0$$

$$\psi_1'(0) = 0 \qquad \psi_2'(0) = 1$$

Fractional Green's function
Constructing solutions of equations with zero RHS

Motivation: consider an integer order example:

$$y''(t) + a^2 y(t) = f(t)$$

$$y(0) = b_1$$

$$y'(0) = b_2$$

Solutions:

$$y(t) = b_1\psi_1(t) + b_2\psi_2(t) + \int\limits_0^t G(t-\tau)f(\tau)d\tau$$

$$= b_1 G'(t) + b_2 G(t) + \int\limits_0^t G(t-\tau)f(\tau)d\tau$$

Fractional Green's function
Constructing solutions of equations with zero RHS

Consider the case of constant coefficients. Then

$$G(t,\tau) \equiv G(t-\tau)$$

Take $\quad y_\lambda(t) = {}_0D_t^\lambda G(t), \quad 0 < \lambda < \sigma_n.$ Then

$${}_0\mathcal{L}_t y_\lambda(t) = {}_0\mathcal{L}_t(\,{}_0D_t^\lambda G(t)) = {}_0D_t^\lambda(\,{}_0\mathcal{L}_t G(t)) = 0.$$

due to zero initial conditions (definition of FGF)

Also, $\qquad {}_0D_t^{\sigma_n-\lambda-1} y_\lambda(t)\Big|_{t=0} = 1.$

$$\begin{aligned}\left[{}_0D_t^{\sigma_n-\lambda-1} y_\lambda(t)\right]_{t=0} &= \left[{}_0D_t^{\sigma_n-\lambda-1}\left({}_0D_t^\lambda G(t)\right)\right]_{t=0} \\ &= \left[{}_0D_t^{\sigma_n-1} G(t)\right]_{t=0} = 1.\end{aligned}$$

Fractional Green's function
Constructing solutions of linear equations with constant coefficients

$$y(t) = \sum_{k=1}^{n} b_k\psi_k(t) + \int\limits_0^t G(t-\tau)f(\tau)d\tau,$$

$$b_k = \left[{}_0\mathcal{D}_t^{\sigma_k-1} y(t)\right]_{t=0}$$

$$\psi_k(t) = {}_0\mathcal{D}_t^{\sigma_n-\sigma_k} G(t), \qquad {}_0\mathcal{D}_t^{\sigma_n-\sigma_k} \equiv {}_aD_t^{\alpha_n}\,{}_aD_t^{\alpha_{n-1}} \ldots {}_aD_t^{\alpha_{k+1}}$$

Fractional Green's function
One-term equation

$$a\,{}_0D_t^\alpha y(t) = f(t)$$

$$g_1(s) = \frac{1}{as^\alpha}$$

$$G_1(t) = \frac{1}{a}\frac{t^{\alpha-1}}{\Gamma(\alpha)}$$

$$y(t) = \frac{1}{a\Gamma(\alpha)}\int\limits_0^t \frac{f(\tau)d\tau}{(t-\tau)^{1-\alpha}} = \frac{1}{a}\,{}_0D_t^{-\alpha} f(t)$$

Fractional Green's function
Two-term equation

$$a\,{}_0D_t^\alpha y(t) + by(t) = f(t)$$

$$g_2(s) = \frac{1}{as^\alpha + b} = \frac{1}{a}\frac{1}{s^\alpha + \frac{b}{a}}$$

$$G_2(t) = \frac{1}{a}t^{\alpha-1}E_{\alpha,\alpha}(-\frac{b}{a}t^\alpha)$$

Fractional Green's function

Three-term equation

$$a\ {}_0D_t^{\beta}y(t) + b\ {}_0D_t^{\alpha}y(t) + c\ y(t) = f(t) \qquad \beta > \alpha$$

$$g_3(s) = \frac{1}{as^{\beta} + bs^{\alpha} + c} = \frac{1}{c}\frac{cs^{-\alpha}}{as^{\beta-\alpha} + b}\frac{1}{1 + \frac{cs^{-\alpha}}{as^{\beta-\alpha}+b}}$$

$$= \frac{1}{c}\sum_{k=0}^{\infty}(-1)^k\left(\frac{c}{a}\right)^{k+1}\frac{s^{-\alpha k-\alpha}}{\left(s^{\beta-\alpha} + \frac{b}{a}\right)^{k+1}}$$

$$G_3(t) = \frac{1}{a}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{c}{a}\right)^k t^{\beta(k+1)-1}E^{(k)}_{\beta-\alpha,\beta+\alpha k}\left(-\frac{b}{a}t^{\beta-\alpha}\right)$$

$$E^{(k)}_{\lambda,\mu}(y) \equiv \frac{d^k}{dy^k}E_{\lambda,\mu}(y) = \sum_{j=0}^{\infty}\frac{(j+k)!\ y^j}{j!\ \Gamma(\lambda j + \lambda k + \mu)}$$

Fractional Green's function

Four-term equation

$$\boxed{\gamma > \beta > \alpha}$$

$$a\ {}_0D_t^{\gamma}y(t) + b\ {}_0D_t^{\beta}y(t) + c\ {}_0D_t^{\alpha}y(t) + d\ y(t) = f(t)$$

$$g_4(p) = \frac{1}{ap^{\gamma} + bp^{\beta} + cp^{\alpha} + d}$$

$$g_4(p) = \frac{1}{ap^{\gamma} + bp^{\beta}}\frac{1}{1 + \frac{cp^{\alpha}+d}{ap^{\gamma}+bp^{\beta}}} = \frac{a^{-1}p^{-\beta}}{p^{\gamma-\beta} + a^{-1}b}\frac{1}{1 + \frac{a^{-1}cp^{\alpha-\beta}+a^{-1}dp^{-\beta}}{p^{\gamma-\beta}+a^{-1}b}}$$

$$= \sum_{m=0}^{\infty}(-1)^m\frac{a^{-1}p^{-\beta}}{(p^{\gamma-\beta}+a^{-1}b)^{m+1}}\left(\frac{c}{a}p^{\alpha-\beta} + \frac{d}{a}p^{-\beta}\right)^m$$

$$= \sum_{m=0}^{\infty}(-1)^m\frac{a^{-1}p^{-\beta}}{(p^{\gamma-\beta}+a^{-1}b)^{m+1}}\sum_{k=0}^{m}\binom{m}{k}\frac{c^k d^{m-k}}{a^m}p^{\alpha k-\beta m}$$

$$= \frac{1}{a}\sum_{m=0}^{\infty}(-1)^m\left(\frac{d}{a}\right)^m\sum_{k=0}^{m}\binom{m}{k}\left(\frac{c}{d}\right)^k\frac{p^{\alpha k-\beta m-\beta}}{(p^{\gamma-\beta}+a^{-1}b)^{m+1}}$$

Fractional Green's function

Four-term equation

$$a\ {}_0D_t^{\gamma}y(t) + b\ {}_0D_t^{\beta}y(t) + c\ {}_0D_t^{\alpha}y(t) + d\ y(t) = f(t)$$

$$g_4(p) = \frac{1}{ap^{\gamma} + bp^{\beta} + cp^{\alpha} + d}$$

$$G_4(t) = \frac{1}{a}\sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{-d}{a}\right)^m\sum_{k=0}^{m}\binom{m}{k}\left(\frac{c}{d}\right)^k t^{\gamma(m+1)-\alpha k-1}E^{(m)}_{\gamma-\beta,\gamma+\beta m-\alpha k}\left(-\frac{b}{a}t^{\gamma-\beta}\right)$$

Fractional Green's function

General case: n-term equation

$$\boxed{\beta_n > \beta_{n-1} > \ldots > \beta_1 > \beta_0}$$

$$a_n\, D^{\beta_n}y(t) + a_{n-1}\, D^{\beta_{n-1}}y(t) + \ldots + a_1\, D^{\beta_1}y(t) + a_0\, D^{\beta_0}y(t) = f(t)$$

$$g_n(p) = \frac{1}{a_np^{\beta_n} + a_{n-1}p^{\beta_{n-1}} + \ldots + a_1p^{\beta_1} + a_0p^{\beta_0}}$$

multinomial coefficients

$$G_n(t) = \frac{1}{a_n}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\sum_{\substack{k_0+k_1+\ldots+k_{n-2}=m\\ k_0\geq 0;\ldots,k_{n-2}\geq 0}}(m;k_0,k_1,\ldots,k_{n-2})$$

$$\prod_{i=0}^{n-2}\left(\frac{a_i}{a_n}\right)^{k_i} t^{(\beta_n-\beta_{n-1})m+\beta_n+\sum_{j=0}^{n-2}(\beta_{n-1}-\beta_j)k_j-1}$$

$$\times E^{(m)}_{\beta_n-\beta_{n-1},+\beta_n+\sum_{j=0}^{n-2}(\beta_{n-1}-\beta_j)k_j}\left(-\frac{a_{n-1}}{a_n}t^{\beta_n-\beta_{n-1}}\right)$$

Power series method

Brook Taylor
(1685 - 1731)

Taylor's expansion:

$$f(t) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}t^n$$

Also: Methodus incrementorum directa et inversa (1715)

- invented integration by parts
- calculus of finite differences
- a change of variables formula
- a way of relating the derivative of a function to the derivative of the inverse function

In 1712 Taylor was appointed to the committee set up to adjudicate on whether the claim of Newton or of Leibniz to have invented the calculus was correct.

Power series method

One term equation: zero initial condition

$${}_0D_t^{\alpha}y(t) = f(t), \quad (t > 0) \qquad \boxed{0 < \alpha < 1}$$

$$y(0) = 0$$

Assume that the RHS can be expanded in Taylor series that converges for $0 \leq t \leq R$:

$$f(t) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}t^n$$

Knowing ${}_0D_t^{\alpha}t^{\nu} = \frac{\Gamma(1+\nu)}{\Gamma(1+\nu-\alpha)}t^{\nu-\alpha}$, we can look for the solution in the form:

$$y(t) = t^{\alpha}\sum_{n=0}^{\infty}y_nt^n = \sum_{n=0}^{\infty}y_nt^{n+\alpha}$$

Power series method

One term equation: zero initial condition

Zero initial conditions are satisfied; consider the equation:

$$ {}_0D_t^{\alpha} y(t) = f(t)$$

$$\sum_{n=0}^{\infty} y_n \frac{\Gamma(1+n+\alpha)}{\Gamma(n+1)} t^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} t^n$$

Comparison of the coefficients gives:

$$y_n = \frac{f^{(n)}(0)}{\Gamma(1+n+\alpha)}, \quad (n = \overline{1,\infty})$$

Therefore, $\quad y(t) = t^{\alpha} \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{\Gamma(1+n+\alpha)} t^n, \quad (0 \le t \le R)$

Power series method

One term equation: non-zero initial condition

$$ {}_0D_t^{\alpha} y(t) = f(t), \quad (t > 0) \qquad \boxed{0 < \alpha < 1}$$

$$y(0) = A, \quad (A \neq 0)$$

The solution exists only if $\quad f(t) \sim \frac{At^{-\alpha}}{\Gamma(1-\alpha)}, \quad (t \to 0)$

Suppose $\quad f(t) = \frac{At^{-\alpha}}{\Gamma(1-\alpha)} + \sum_{n=1}^{\infty} f_n t^{n-\alpha} \quad$ (coeffs. are known)

and look for the solution in the form

$$y(t) = \sum_{n=0}^{\infty} y_n t^n.$$

Power series method

One term equation: zero initial condition

In the considered simple case we have:

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \frac{\Gamma(n+1)}{\Gamma(1+n+\alpha)} t^{n+\alpha} \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} {}_0D_t^{-\alpha} t^n \\ &= {}_0D_t^{-\alpha} \left\{ \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} t^n \right\} \\ &= {}_0D_t^{-\alpha} f(t). \end{aligned}$$

Power series method

One term equation: non-zero initial condition

Consider the equation:

$$ {}_0D_t^{\alpha} y(t) = f(t)$$

$$\sum_{n=0}^{\infty} y_n \frac{\Gamma(1+n)}{\Gamma(a+n-\alpha)} t^{n-\alpha} = \frac{At^{-\alpha}}{\Gamma(1-\alpha)} + \sum_{n=1}^{\infty} f_n t^{n-\alpha}$$

Comparison of the coefficients gives:

$$y_0 = A; \quad y_n = \frac{\Gamma(1+n-\alpha)}{\Gamma n + 1} f_n, \quad (n = \overline{1,\infty})$$

$$y(t) = \sum_{n=0}^{\infty} y_n t^n.$$

Power series method

One term equation: weak singularity in the RHS

Suppose $\quad f(t) = t^{\beta} g(t), \quad (\beta > -1), \qquad g(t) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} t^n.$

and $\quad \alpha + \beta > 0$

Look for the solution in the form:

$$y(t) = t^{\alpha+\beta} \sum_{n=0}^{\infty} y_n t^n = \sum_{n=0}^{\infty} y_n t^{n+\alpha+\beta}$$

Then the coefficients in the solution are:

$$y_n = \frac{\Gamma(1+n+\beta)\, g^{(n)}(0)}{\Gamma(1+n+\alpha+\beta)\,\Gamma(n+1)}, \quad (n = \overline{1,\infty}).$$

Power series method

One term equation: initial condition in terms of R-L integral

$$ {}_0D_t^{\alpha} y(t) = f(t), \quad (t > 0) \qquad \boxed{0 < \alpha < 1}$$

$$ {}_0D_t^{\alpha-1} y(t) \Big|_{t=0} = B$$

Suppose that the RHS can be expanded in Taylor series:

$$f(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} t^n$$

We can look for the solution in the form:

$$y(t) = t^{\alpha-1} \sum_{n=0}^{\infty} y_n t^n = \sum_{n=0}^{\infty} y_n t^{n+\alpha-1}$$

Power series method

One term equation: initial condition in terms of R-L integral

Consider the equation:

$${}_0D_t^{\alpha} y(t) = f(t)$$

$$\sum_{n=0}^{\infty} y_{n+1}\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}t^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{\Gamma(n+1)}t^n$$

$$\boxed{1/\Gamma(0) = 0}$$

Comparison of the coefficients gives:

$$y_{n+1} = \frac{f^{(n)}(0)}{\Gamma(n+\alpha+1)}, \quad (n = \overline{0,\infty})$$

Still have to determine y_0

Power series method

One term equation: initial condition in terms of R-L integral

To determine y_0, the initial condition must be used.

$$\begin{aligned} {}_0D_t^{\alpha-1} y(t) &= \sum_{n=0}^{\infty} y_n \, {}_0D_t^{\alpha-1} t^{n+\alpha+1} \\ &= \sum_{n=0}^{\infty} y_n \frac{\Gamma(n+\alpha)}{\Gamma(n+1)} t^n \end{aligned}$$

$$\boxed{{}_0D_t^{\alpha} t^{\nu} = \frac{\Gamma(1+\nu)}{\Gamma(1+\nu-\alpha)}t^{\nu-\alpha}}$$

Taking the limit as $t \to 0$ we obtain:

$${}_0D_t^{\alpha-1} y(t)\Big|_{t=0} = \Gamma(\alpha)\, y_0 \quad \Rightarrow \quad y_0 = \frac{B}{\Gamma(\alpha)}$$

Power series method

- Examine the initial conditions and the RHS
- Initial conditions and the RHS of the equation determine the class of solutions and dictate the form of the series
- The key formula is

$${}_0D_t^{\alpha} t^{\nu} = \frac{\Gamma(1+\nu)}{\Gamma(1+\nu-\alpha)}t^{\nu-\alpha}$$

Power series method

Equation with non-constant coefficients

Consider the following problem:

$$\frac{d}{dt}\Big(f(t)\,(y(t)+1)\Big) + \lambda \, {}_0D_t^{1/2} y(t) = 0, \qquad (0 < t < 1)$$

$$y(0) = 0,$$

For some particular types of $f(t)$ a solution can be obtained. Suppose

$$f(t) = \sum_{n=0}^{\infty} f_n t^{n/2}, \quad f_0 = 1.$$

Then the solution can have the form:

$$y(t) = \sum_{n=1}^{\infty} y_n t^{n/2}$$

Power series method

Equation with non-constant coefficients

Initial condition is satisfied by the chosen form of solution.

$$\boxed{f(t) = \sum_{n=0}^{\infty} f_n t^{n/2}, \quad f_0 = 1} \qquad \boxed{y(t) = \sum_{n=1}^{\infty} y_n t^{n/2}}$$

$$\frac{d}{dt}\Big(f(t)\,(y(t)+1)\Big) + \lambda \, {}_0D_t^{1/2} y(t) = 0$$

The the comparison of the coefficients gives:

$$y_1 = -f_1, \qquad \sum_{k=0}^{n} y_{n+1-k} f_k + \lambda y_n \frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n+3}{2}\right)} = -f_{n+1}$$

recurrence

Power series method

Equation with non-constant coefficients: particular case

$$\boxed{y_1 = -f_1, \qquad \sum_{k=0}^{n} y_{n+1-k} f_k + \lambda y_n \frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n+3}{2}\right)} = -f_{n+1}}$$

Take, for example,

$$f(t) = 1 - \sqrt{t}$$

Then

$$\begin{aligned} y_1 &= 1, \\ y_2 &= y_1\left(1 - \lambda\frac{\Gamma(3/2)}{\Gamma(2)}\right), \\ &\dots \\ y_{n+1} &= y_n\left(1 - \lambda\frac{\Gamma\left(\frac{n+2}{2}\right)}{\Gamma\left(\frac{n+3}{2}\right)}\right) \end{aligned}$$

Power series method

Equation with non-constant coefficients: even more particular case

If we take

$$\lambda = \frac{\Gamma\left(\frac{n+2}{2}\right)}{\Gamma\left(\frac{n+3}{2}\right)}$$

then

$$\lambda = \frac{\Gamma(2)}{\Gamma(3/2)} = \frac{2}{\sqrt{\pi}}, \qquad y(t) = \sqrt{t}.$$

$$\lambda = \frac{\Gamma(5/2)}{\Gamma(2)} = \frac{3\sqrt{\pi}}{4}, \qquad y(t) = \sqrt{t} + \left(1 - \frac{3\pi}{8}\right) t.$$

Power series method

Two-term nonlinear equation

Consider the following problem:

λ and y_0 are given constants

$$_0D_t^{1/2} y(t) - \lambda\Big(y(t) - y_0\Big)^2 = 0, \quad (t > 0)$$

$$y(0) = 0$$

The solution can have the form:

$$y(t) = \sum_{n=1}^{\infty} y_n t^{n/2}$$

(because $_0D_t^{1/2} y(t)$ and $(y - A)^2$ both give the series of the same form)

Power series method

Two-term nonlinear equation

Initial condition is satisfied; use the equation:

$$y(t) = \sum_{n=1}^{\infty} y_n t^{n/2}$$

$$_0D_t^{1/2} y(t) - \lambda\Big(y(t) - y_0\Big)^2 = 0$$

and we obtain the recurrence relationships:

$$\begin{aligned} y_1 &= \lambda\, y_0^2 \frac{\Gamma(1)}{\Gamma(\frac{3}{2})}, & y_3 &= \lambda\,(y_1^2 + 2\, y_0\, y_2)\frac{\Gamma(2)}{\Gamma(\frac{5}{2})}, \\ & & \ldots &\;\ldots\;\ldots \\ y_2 &= 2\,\lambda\, y_0\, y_1 \frac{\Gamma(\frac{3}{2})}{\Gamma(2)}, & y_n &= \lambda \sum_{k=0}^{n-1} y_k\, y_{n-k-1} \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n+2}{2})}. \end{aligned}$$