

The Laplace transform method for fractional differential equations

1

The Laplace transform method for FDEs

EXAMPLE 2.

$${}_0D_t^Q f(t) + {}_0D_t^q f(t) = h(t), \quad 0 < q < Q < 1$$

Solution: applying the Laplace transform:

$$(s^Q + s^q)F(s) = C + H(s), \quad C = [{}_0D_t^{q-1} f(t) + {}_0D_t^{Q-1} f(t)]_{t=0}$$

$$F(s) = \frac{C + H(s)}{s^Q + s^q} = \frac{C + H(s)}{s^q(s^{Q-q} + 1)} = (C + H(s)) \frac{s^{-q}}{s^{Q-q} + 1}.$$

and the inverse transform gives the solution:

$$f(t) = C G(t) + \int_0^t G(t-\tau) h(\tau) d\tau,$$

$$C = [{}_0D_t^{q-1} f(t) + {}_0D_t^{Q-1} f(t)]_{t=0}, \quad G(t) = t^{Q-1} E_{Q-q, Q}(-t^{Q-q})$$

4

The Laplace transform method for fractional differential equations

$$\int_0^\infty e^{-st} {}_0D_t^\alpha f(t) dt = s^\alpha F(s) - \sum_{k=0}^{n-1} s^k [{}_0D_t^{\alpha-k-1} f(t)]_{t=0},$$

(n-1 < α ≤ n).

$$t^{\alpha k + \beta - 1} E_{\alpha, \beta}^{(k)}(\pm zt^\alpha) \Leftrightarrow \frac{k! p^{\alpha-\beta}}{(p^\alpha \mp a)^{k+1}}$$

2

The Laplace transform method for FDEs

EXAMPLE 3.

$${}_0D_t^\alpha y(t) - \lambda y(t) = h(t), \quad (t > 0); \quad n-1 < \alpha < n.$$

$$[{}_0D_t^{\alpha-k} y(t)]_{t=0} = b_k, \quad (k = 1, 2, \dots, n)$$

Solution: applying the Laplace transform:

$$s^\alpha Y(s) - \lambda Y(s) = H(s) + \sum_{k=1}^n b_k s^{k-1},$$

$$Y(s) = \frac{H(s)}{s^\alpha - \lambda} + \sum_{k=1}^n b_k \frac{s^{k-1}}{s^\alpha - \lambda}$$

and the inverse transform gives the solution:

$$y(t) = \sum_{k=1}^n b_k t^{\alpha-k} E_{\alpha, \alpha-k+1}(\lambda t^\alpha) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha, \alpha}(\lambda(t-\tau)^\alpha) h(\tau) d\tau$$

5

The Laplace transform method for FDEs

EXAMPLE 1.

$${}_0D_t^{1/2} f(t) + af(t) = 0, \quad (t > 0); \quad [{}_0D_t^{-1/2} f(t)]_{t=0} = C$$

Solution: applying the Laplace transform we obtain

$$F(s) = \frac{C}{s^{1/2} + a}, \quad C = [{}_0D_t^{-1/2} f(t)]_{t=0}$$

and the inverse Laplace transform gives the solution:

$$f(t) = Ct^{-1/2} E_{\frac{1}{2}, \frac{1}{2}}(-a\sqrt{t}).$$

If $a=1$, then

$$f(t) = C(\frac{1}{\sqrt{\pi t}} - e^t \operatorname{erfc}(\sqrt{t}))$$

3

The Laplace transform method for fractional differential equations with sequential derivatives

$$L\{ {}_0D_t^{\sigma_m} f(t); s \} = s^{\sigma_m} F(s) - \sum_{k=0}^{m-1} s^{\sigma_m - \sigma_{m-k}} [{}_0D_t^{\sigma_{m-k}-1} f(t)]_{t=0},$$

$${}_aD_t^{\sigma_{m-k}-1} \equiv {}_aD_t^{\alpha_{m-k}-1} {}_aD_t^{\alpha_{m-k}-1} \dots {}_aD_t^{\alpha_1},$$

(k = 0, 1, ..., m-1).

$$t^{\alpha k + \beta - 1} E_{\alpha, \beta}^{(k)}(\pm zt^\alpha) \Leftrightarrow \frac{k! p^{\alpha-\beta}}{(p^\alpha \mp a)^{k+1}}$$

6

The Laplace transform method for SqFDEs

EXAMPLE 4.

$${}_0D_t^\alpha \left({}_0D_t^\beta y(t) \right) + ay(t) = 0$$

$$\left[{}_0D_t^{\alpha-1} \left({}_0D_t^\beta y(t) \right) \right]_{t=0} = b_1, \quad \left[{}_0D_t^{\beta-1} y(t) \right]_{t=0} = b_2,$$

$$0 < \alpha < 1, \quad 0 < \beta < 1, \quad \alpha + \beta = 1/2.$$

Solution: applying the Laplace transform:

$$(s^{\alpha+\beta} + a)Y(s) = s^\alpha b_2 + b_1,$$

$$Y(s) = b_2 \frac{s^\alpha}{s^{\alpha+\beta} + a} + b_1 \frac{1}{s^{\alpha+\beta} + a}$$

$$\sigma_1 = \beta, \quad \sigma_2 = \alpha, \quad \text{and } m = 2$$

$$\sigma_1 = \beta, \quad \sigma_2 = \alpha + \beta$$

and the inverse transform gives the solution:

$$y(t) = b_2 t^{\beta-1} E_{\alpha+\beta, \beta}(-at^{\alpha+\beta}) + b_1 t^{\alpha+\beta-1} E_{\alpha+\beta, \alpha+\beta}(-at^{\alpha+\beta})$$

For $\beta = 0$ and $\alpha = 1/2$ (and, of course, $b_2 = 0$) we have the solution of Example 1.

7

To recall the standard procedure:

$$\frac{dY(t)}{dt} + aY(t) = bF(t)$$

$$sy(s) - Y(0^+) + ay(s) = bf(s)$$

$$y(s) = \frac{Y(0^+)}{s+a} + \left[\frac{b}{s+a} \right] f(s)$$

$$Y(t) = Y(0^+) e^{-at} + b \int_0^t e^{-a(t-\tau)} F(\tau) d\tau.$$

10

The Laplace transform method for SqFDEs

EXAMPLE 5.

$${}_0D_t^\alpha \left({}_0D_t^\beta y(t) \right) + {}_0D_t^q y(t) = h(t),$$

$$0 < \alpha < 1, 0 < \beta < 1, 0 < q < 1, \alpha + \beta = Q > q.$$

Solution: applying the Laplace transform:

$$(s^{\alpha+\beta} + s^q)Y(s) = H(s) + s^\alpha b_2 + b_1,$$

$$b_1 = \left[{}_0D_t^{q-1} \left({}_0D_t^\beta y(t) \right) \right]_{t=0} + \left[{}_0D_t^{q-1} y(t) \right]_{t=0}, \quad b_2 = \left[{}_0D_t^{\beta-1} y(t) \right]_{t=0}$$

$$Y(s) = \frac{s^{-q} H(s)}{s^{\alpha+\beta-q} + 1} + b_2 \frac{s^{\alpha-q}}{s^{\alpha+\beta-q} + 1} + b_1 \frac{s^{-q}}{s^{\alpha+\beta-q} + 1}$$

and the inverse transform gives the solution:

$$y(t) = b_2 t^{\beta-1} E_{\alpha+\beta-q, \beta}(-t^{\alpha+\beta-q}) + b_1 t^{\alpha+\beta-q} E_{\alpha+\beta-q, \alpha+\beta}(-t^{\alpha+\beta-q})$$

$$+ \int_0^t (t-\tau)^{\alpha+\beta-1} E_{\alpha+\beta-q, \alpha+\beta}(-(t-\tau)^{\alpha+\beta-q}) h(\tau) d\tau.$$

8

Problem #1

Find a general solution of the FDE with R-L derivative:

$${}_0D_t^{1/2} Y(t) + aY(t) = bF(t).$$

Solution: $\sqrt{s} y(s) - \left[{}_0D_t^{-1/2} Y(t) \right]_{t=0} + ay(s) = bf(s)$

$$y(s) = \frac{c}{\sqrt{s+a}} + \left[\frac{b}{\sqrt{s+a}} \right] f(s) \quad c = \left[{}_0D_t^{-1/2} Y(t) \right]_{t=0}$$

If $F(t) = 0$, then $Y(t) = \frac{c}{\sqrt{t}} E_{1/2, 1/2}(-a\sqrt{t})$

In general, $Y(t) = \frac{c}{\sqrt{t}} E_{1/2, 1/2}(-a\sqrt{t}) + b \int_0^t \frac{E_{1/2, 1/2}(-a\sqrt{t-\tau})}{\sqrt{t-\tau}} F(\tau) d\tau$

11

Do it yourself !

9

Problem # 2

Find a general solution of the FDE with Caputo derivative:

$${}_0D_t^{1/2} Y(t) + a Y(t) = b F(t)$$

Solution: $\sqrt{s} y(s) - \frac{1}{\sqrt{s}} [Y(0^+)] + ay(s) = bf(s)$

$$y(s) = \frac{[Y(0^+)]}{\sqrt{s} (\sqrt{s+a})} + \frac{bf(s)}{\sqrt{s+a}}$$

If $F(t) = 0$, then

$$Y(t) = A E_{1/2, 1}[-a\sqrt{t}] \quad A = Y(0^+)$$

In general case: similar to Problem #1 (+ convolution)

12

Problem # 3

From:
Magin R, Fractional Calculus in Bioengineering,
Begell House Inc., Redding 2006, p. 151.

Find a solution of the FDE with R-L derivative:

$$\frac{dF(t)}{dt} + \frac{d^{-1/2}F(t)}{dt^{-1/2}} - 2F(t) = 0$$

with initial conditions:

$$F(0^+) = 0, \quad \frac{d^{-1/2}F(0^+)}{dt^{-1/2}} = c.$$

Hint: $s + \sqrt{s} - 2 = (\sqrt{s} - 1)(\sqrt{s} + 2)$

Formal solution: $F(t) = \frac{c}{3\sqrt{t}} [E_{1/2,1/2}(-2\sqrt{t}) - E_{1/2,1/2}(\sqrt{t})]$

Question: what can you say about the initial conditions?

13

Problem # 4

From:
Magin R, Fractional Calculus in Bioengineering,
Begell House Inc., Redding 2006, p. 152.

Solve the following initial value problems:

(a) ${}_0D_t^{1/2}Y(t) - Y(t) = -1 \quad (b) \frac{dY(t)}{dt} - Y(t) = -1 - \frac{1}{\sqrt{\pi t}}$
 $[{}_0D_t^{-1/2}Y(t)]_{t=0} = 0 \quad Y(0) = 0$

Explain the obtained results. Click when you are ready.

The solution in both cases is: $Y(t) = 1 - E_{1/2}(\sqrt{t}).$
Equation (a) can be transformed to equation (b).

14

Problem # 9

Use the Miller-Ross formula

$$\frac{1}{s^\alpha - a} = \sum_{j=1}^q \frac{a^{j-1}}{s^{j\alpha-1}(s-a^q)}, \quad \text{where } \alpha = 1/q, \quad q \in N$$

for evaluation of the inverse Laplace transforms:

(a) $L^{-1}\left\{\frac{1}{\sqrt{s-a}}\right\} \quad (b) L^{-1}\left\{\frac{1}{\sqrt[3]{s-a}}\right\} \quad (c) L^{-1}\left\{\frac{1}{\sqrt{s}(\sqrt{s}-a)}\right\}$

Answer: (a) $\frac{1}{\sqrt{t}}E_{1,1/2}(a^2t) + aE_{1,1}(a^2t)$

(b) $t^{-2/3}E_{1,1/3}(a^3t) + at^{-1/3}E_{1,2/3}(a^3t) + a^2E_{1,1}(a^3t)$

(c) $a\sqrt{t}E_{1,3/2}(a^2t) + E_{1,1}(a^2t)$

15