Integral transforms of fractional derivatives

(continued...)

Mellin transform — basic facts

$f(t)$ is piece-wise continuous in every closed interval $[a, b] \subset (0, \infty)$

$$F(s) = \mathcal{M}\Big\{f(t);\ s\Big\} = \int_0^\infty f(t) t^{s-1} dt, \qquad \gamma_1 < Re(s) < \gamma_2.$$

Inverse Mellin transform:

$$f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} F(s) t^{-s} ds, \quad (0 < t < \infty), \qquad \gamma_1 < \gamma < \gamma_2.$$

(under the Dirichlet conditions in every closed interval $[a, b] \subset (0, \infty)$:
$f(t)$ must have a finite number of extrema,
must have a finite number of discontinuities,
must be absolutely integrable.)

Hjalmar Mellin was G. M. Mittag-Leffler's student at the university in Helsinki

Mellin transform — basic facts

From the definition we have:

$$\mathcal{M}\Big\{t^\alpha f(t);\ s\Big\} = \mathcal{M}\Big\{f(t);\ s + \alpha\Big\} = F(s + \alpha).$$

Mellin convolution:

$$f(t) * g(t) = \int_0^\infty f(t\tau) g(\tau) d\tau$$

$$\mathcal{M}\Big\{\int_0^\infty f(t\tau) g(\tau) d\tau\,;\ s\Big\} = F(s)\, G(1 - s)$$

Combining the above formulas:

$$\mathcal{M}\Big\{t^\lambda \int_0^\infty \tau^\mu f(t\tau) g(\tau) d\tau\,;\ s\Big\} = F(s + \lambda)\, G(1 - s - \lambda + \mu)$$

Mellin transform — basic facts

Mellin transform of integer order derivatives

Integrating repeatedly by parts:

$$\begin{aligned}
\mathcal{M}\Big\{f^{(n)}(t);\ s\Big\} &= \int_0^\infty f^{(n)}(t)\, t^{s-1} dt \\
&= \Big[f^{(n-1)}(t)\, t^{s-1}\Big]_0^\infty - (s-1) \int_0^\infty f^{(n-1)}(t)\, t^{s-2} dt \\
&= \Big[f^{(n-1)}(t)\, t^{s-1}\Big]_0^\infty - (s-1) \mathcal{M}\Big\{f^{(n-1)}(t);\ s - 1\Big\} \\
&= \ldots \\
&= \sum_{k=0}^{n-1} (-1)^k \frac{\Gamma(s)}{\Gamma(s-k)} \Big[f^{(n-k-1)}(t)\, t^{s-k-1}\Big]_0^\infty \\
&\qquad + (-1)^n \frac{\Gamma(s)}{\Gamma(s-n)} F(s-n) \\
&= \sum_{k=0}^{n-1} \frac{\Gamma(1-s+k)}{\Gamma(1-s)} \Big[f^{(n-k-1)}(t)\, t^{s-k-1}\Big]_0^\infty \\
&\qquad + \frac{\Gamma(1-s+n)}{\Gamma(1-s)} F(s-n)
\end{aligned}$$

If $f(t)$ and $Re(s)$ turns all substitutions into zero, then

$$\mathcal{M}\Big\{f^{(n)}(t);\ s\Big\} = \frac{\Gamma(1-s+n)}{\Gamma(1-s)} F(s-n)$$

Mellin transform of the R-L integral

$${}_0D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau \overset{\boxed{\tau = t\xi}}{=} \frac{t^\alpha}{\Gamma(\alpha)} \int_0^1 (1-\xi)^{\alpha-1} f(t\xi) d\xi = \frac{t^\alpha}{\Gamma(\alpha)} \int_0^\infty f(t\xi) g(\xi) d\xi$$

where $g(t) = \begin{cases} (1-t)^{\alpha-1}, & (0 \le t < 1) \\ 0, & (t \ge 1) \end{cases}$

The Mellin transform of *g(t)* is the Beta function:

$$\mathcal{M}\Big\{g(t);\ s\Big\} = B(\alpha, s) = \frac{\Gamma(\alpha)\Gamma(s)}{\Gamma(\alpha + s)}$$

Mellin transform of the R-L integral

We already have:

$$\mathcal{M}\Big\{t^\lambda \int_0^\infty \tau^\mu f(t\tau) g(\tau) d\tau\,;\ s\Big\} = F(s + \lambda)\, G(1 - s - \lambda + \mu)$$

$${}_0D_t^{-\alpha} f(t) = \frac{t^\alpha}{\Gamma(\alpha)} \int_0^\infty f(t\xi) g(\xi) d\xi$$

$$\mathcal{M}\Big\{g(t);\ s\Big\} = B(\alpha, s) = \frac{\Gamma(\alpha)\Gamma(s)}{\Gamma(\alpha + s)}$$

$$\mathcal{M}\Big\{{}_0D_t^{-\alpha} f(t);\ s\Big\} = \frac{1}{\Gamma(\alpha)} F(s + \alpha)\, B(\alpha, 1 - s - \alpha)$$

$$\boxed{\mathcal{M}\Big\{{}_0D_t^{-\alpha} f(t);\ s\Big\} = \frac{\Gamma(1 - s - \alpha)}{\Gamma(1 - s)} F(s + \alpha)}$$

Compare with

$$\mathcal{M}\Big\{f^{(n)}(t);\ s\Big\} = \frac{\Gamma(1-s+n)}{\Gamma(1-s)} F(s-n)$$

Mellin transform of the R-L derivative

Let us take $0 \le n-1 < \alpha < n$. Then we have: ${}_0D_t^{\alpha} f(t) = \frac{d^n}{dt^n}\, {}_0D_t^{-(n-\alpha)} f(t)$.

$$\mathcal{M}\Big\{ {}_0D_t^{\alpha} f(t);\ s\Big\} = \mathcal{M}\Big\{ \frac{d^n}{dt^n}\, {}_0D_t^{\alpha-n} f(t);\ s\Big\} = \mathcal{M}\Big\{ g^{(n)};\ s\Big\}$$

$$g(t) = {}_0D_t^{-(n-\alpha)} f(t)$$

$$= \sum_{k=0}^{n-1} \frac{\Gamma(1-s+k)}{\Gamma(1-s)} \Big[g^{(n-k-1)}(t)\, t^{s-k-1} \Big]_0^{\infty} + \frac{\Gamma(1-s+n)}{\Gamma(1-s)} G(s-n)$$

$$= \sum_{k=0}^{n-1} \frac{\Gamma(1-s+k)}{\Gamma(1-s)} \left[\frac{d^{n-k-1}}{dt^{n-k-1}}\, {}_0D_t^{\alpha-n} f(t)\, t^{s-k-1} \right]_0^{\infty} + \frac{\Gamma(1-s+n)}{\Gamma(1-s)} \frac{\Gamma(1-(s-n)-(n-\alpha))}{\Gamma(1-(s-n))} F((s-n)+(n-\alpha))$$

or, after simplifications:

$$\boxed{\mathcal{M}\Big\{ {}_0D_t^{\alpha} f(t);\ s\Big\} = \sum_{k=0}^{n-1} \frac{\Gamma(1-s+k)}{\Gamma(1-s)} \Big[{}_0D_t^{\alpha-k-1} f(t)\, t^{s-k-1} \Big]_0^{\infty} + \frac{\Gamma(1-s+\alpha)}{\Gamma(1-s)} F(s-\alpha)}$$

Mellin transform of the R-L derivative

We have obtained:

$$\mathcal{M}\Big\{ {}_0D_t^{\alpha} f(t);\ s\Big\} = \sum_{k=0}^{n-1} \frac{\Gamma(1-s+k)}{\Gamma(1-s)} \Big[{}_0D_t^{\alpha-k-1} f(t)\, t^{s-k-1} \Big]_0^{\infty} + \frac{\Gamma(1-s+\alpha)}{\Gamma(1-s)} F(s-\alpha)$$

notice this

If $0 < \alpha < 1$, then:

$$\mathcal{M}\Big\{ {}_0D_t^{\alpha} f(t);\ s\Big\} = \Big[{}_0D_t^{\alpha-1} f(t)\, t^{s-1} \Big]_0^{\infty} + \frac{\Gamma(1-s+\alpha)}{\Gamma(1-s)} F(s-\alpha)$$

If $f(t)$ and $Re(s)$ turn all substitutions into zero, then

$$\mathcal{M}\Big\{ {}_0D_t^{\alpha} f(t);\ s\Big\} = \frac{\Gamma(1-s+\alpha)}{\Gamma(1-s)} F(s-\alpha).$$

similar to the MT of R-L integral

Mellin transform of the Caputo derivative

Take $0 \le n-1 < \alpha < n$ and denote $h(t) = f^{(n)}(t)$. Then:

$$\mathcal{M}\Big\{ {}_0^CD_t^{\alpha} f(t);\ s\Big\} = \mathcal{M}\Big\{ {}_0D_t^{-(n-\alpha)} f^{(n)}(t);\ s\Big\} = \mathcal{M}\Big\{ {}_0D_t^{-(n-\alpha)} h(t);\ s\Big\} = \frac{1-s-(n-\alpha)}{\Gamma(1-s)} H(s+(n-\alpha))$$

$$= \sum_{k=0}^{n-1} \frac{\Gamma(1-s-n+\alpha+k)}{\Gamma(1-s)} \Big[f^{(n-k-1)}(t)\, t^{s+n-\alpha-k-1} \Big]_0^{\infty} + \frac{\Gamma(1-s+\alpha)}{\Gamma(1-s)} F(s-\alpha)$$

notice this

If $0 < \alpha < 1$, then:

$$\mathcal{M}\Big\{ {}_0^CD_t^{\alpha} f(t);\ s\Big\} = \frac{\Gamma(\alpha-s)}{\Gamma(1-s)} \Big[f(t)\, t^{s-\alpha} \Big]_0^{\infty} + \frac{\Gamma(1-s+\alpha)}{\Gamma(1-s)} F(s-\alpha)$$

If $f(t)$ and $Re(s)$ turn all substitutions into zero, then

$$\mathcal{M}\Big\{ {}_0^CD_t^{\alpha} f(t);\ s\Big\} = \frac{\Gamma(1-s+\alpha)}{\Gamma(1-s)} F(s-\alpha)$$

similar to the MT of R-L integral and derivative

Mellin transform of sequential derivatives

Recall notation:

$${}_a\mathcal{D}_t^{\sigma_m} \equiv {}_aD_t^{\alpha_m}\, {}_aD_t^{\alpha_{m-1}} \ldots {}_aD_t^{\alpha_1};$$

$${}_a\mathcal{D}_t^{\sigma_m - 1} \equiv {}_aD_t^{\alpha_m - 1}\, {}_aD_t^{\alpha_{m-1}} \ldots {}_aD_t^{\alpha_1};$$

$$\sigma_m = \sum_{j=1}^{m} \alpha_j, \qquad 0 < \alpha_j \le 1, \quad (j = 1, 2, \ldots, m)$$

It can be shown that

$$\mathcal{M}\Big\{ {}_0\mathcal{D}_t^{\sigma_n} f(t);\ s\Big\} = \sum_{k=1}^{n} \frac{\Gamma(1-s+\sigma_n-\sigma_k)}{\Gamma(1-s)} \Big[{}_0\mathcal{D}_t^{\sigma_k - 1} f(t)\, t^{s-\sigma_n+\sigma_k-1} \Big]_0^{\infty} + \frac{\Gamma(1-s+\sigma_n)}{\Gamma(1-s)} F(s-\sigma_n)$$

If $f(t)$ and $Re(s)$ turn all substitutions into zero, then

$$\mathcal{M}\Big\{ {}_0\mathcal{D}_t^{\sigma_n} f(t);\ s\Big\} = \frac{\Gamma(1-s+\sigma_n)}{\Gamma(1-s)} F(s-\sigma_n)$$

For functions with suitable behavior the Mellin transforms for R-L, Caputo, and M-R derivatives coincide.

Useful property of the Mellin transform

For functions with suitable behavior we have:

$$\mathcal{M}\Big\{ t^{\alpha} D^{\alpha} f(t);\ s\Big\} = \frac{\Gamma(1-s)}{\Gamma(1-s-\alpha)} F(s)$$

Then

$$\mathcal{M}\Big\{ \sum_{k=0}^{n} a_k t^{\alpha+k} D^{\alpha+k} f(t);\ s\Big\} = F(s)\Gamma(1-s) \sum_{k=0}^{n} \frac{a_k}{\Gamma(1-s-\alpha-k)}$$

$$= \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(1-s-\alpha)} \sum_{k=0}^{n} (-1)^k a_k \prod_{j=0}^{k-1} (s+\alpha+1)$$

In particular,

$$\mathcal{M}\Big\{ t^{\alpha+1} D^{\alpha+1} f(t) + t^{\alpha} D^{\alpha} f(t);\ s\Big\} = \frac{\Gamma(1-s)(1-s-\alpha)}{\Gamma(1-s-\alpha)} F(s)$$

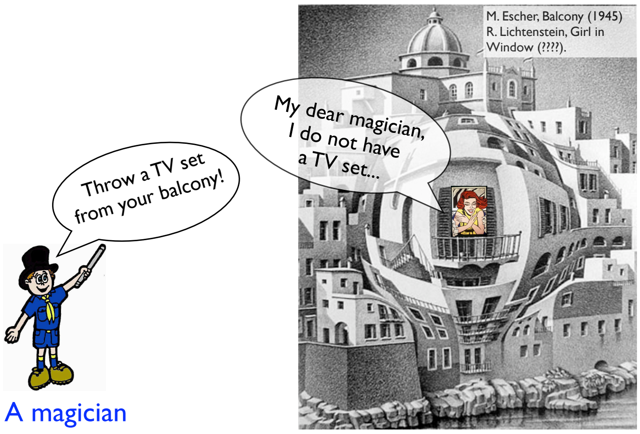
and for $\alpha = 1$ we have the well known property of the Mellin transform:

$$\mathcal{M}\Big\{ t^2 f''(t) + t f'(t);\ s\Big\} = s^2 F(s)$$

often used in applications

Fractional Differential Equations

Existence and uniqueness of solutions



A magician

Existence and uniqueness of solutions
Linear fractional differential equations

We shall consider the following IVP:

$$ {}_0\mathcal{D}_t^{\sigma_n} y(t) + \sum_{j=1}^{n-1} p_j(t)\, {}_0\mathcal{D}_t^{\sigma_{n-j}} y(t) + p_n(t) y(t) = f(t), \qquad (0 < t < T < \infty) $$

$$ \left[{}_0\mathcal{D}_t^{\sigma_k - 1} y(t) \right]_{t=0} = b_k, \qquad k = 1, \dots, \; n $$

where ${}_a\mathcal{D}_t^{\sigma_k} \equiv {}_aD_t^{\alpha_k}\, {}_aD_t^{\alpha_{k-1}} \dots {}_aD_t^{\alpha_1}$;

${}_a\mathcal{D}_t^{\sigma_k - 1} \equiv {}_aD_t^{\alpha_k - 1}\, {}_aD_t^{\alpha_{k-1}} \dots {}_aD_t^{\alpha_1}$;

$\sigma_k = \sum_{j=1}^{k} \alpha_j, \quad (k = 1, 2, \dots, n);$

$0 < \alpha_j \le 1, \quad (j = 1, 2, \dots, n),$

and $f(t) \in L_1(0, T)$, i.e.

$$ \int_0^T |f(t)| dt < \infty. $$

Existence and uniqueness of solutions
The simplest equation

THEOREM 1. *If $f(t) \in L_1(0, T)$, then the equation*

$$ {}_0\mathcal{D}_t^{\sigma_n} y(t) = f(t) $$

has the unique solution $y(t) \in L_1(0, T)$, which satisfies the initial conditions

$$ \left[{}_0\mathcal{D}_t^{\sigma_k - 1} y(t) \right]_{t=0} = b_k, \qquad k = 1, \dots, \; n $$

PROOF. Let us construct the solution and then prove its uniqueness.

Existence and uniqueness of solutions
The simplest equation

Constructing a solution:

The Laplace transform of the sequential FD gives

$$ s^{\sigma_n} Y(s) - \sum_{k=0}^{n-1} s^{\sigma_n - \sigma_{n-k}} \left[{}_0\mathcal{D}_t^{\sigma_{n-k} - 1} y(t) \right]_{t=0} = F(s) $$

$$ Y(s) = s^{-\sigma_n} F(s) + \sum_{k=0}^{n-1} b_{n-k} s^{-\sigma_{n-k}} $$

Inverse Laplace transform gives

$$ \begin{aligned} y(t) &= \frac{1}{\Gamma(\sigma_n)} \int_0^t (t-\tau)^{\sigma_n - 1} f(\tau)\, d\tau + \sum_{k=0}^{n-1} \frac{b_{n-k}}{\Gamma(\sigma_{n-k})} t^{\sigma_{n-k} - 1} \\ &= \frac{1}{\Gamma(\sigma_n)} \int_0^t (t-\tau)^{\sigma_n - 1} f(\tau)\, d\tau + \sum_{i=1}^{n} \frac{b_i}{\Gamma(\sigma_i)} t^{\sigma_i - 1} \end{aligned} \qquad \boxed{i = n - k} $$

Existence and uniqueness of solutions
The simplest equation

Checking if it is a solution:

Using the R-L derivative of the power function and taking into account that $\frac{1}{\Gamma(-m)} = 0, \qquad m = 0, 1, 2, \dots,$

we obtain

$$ {}_0\mathcal{D}_t^{\sigma_k} \left(\frac{t^{\sigma_i - 1}}{\Gamma(\sigma_i)} \right) = \begin{cases} \dfrac{t^{\sigma_i - \sigma_k - 1}}{\Gamma(\sigma_i - \sigma_k)}, & (k < i) \\ 0, & (k \ge i) \end{cases} $$

$$ {}_0\mathcal{D}_t^{\sigma_k - 1} \left(\frac{t^{\sigma_i - 1}}{\Gamma(\sigma_i)} \right) = \begin{cases} \dfrac{t^{\sigma_i - \sigma_k}}{\Gamma(1 + \sigma_i - \sigma_k)}, & (k < i) \\ 1, & (k = i) \\ 0, & (k > i) \end{cases} $$

Direct substitution of y(t) into the equation and the initial conditions shows that they are satisfied. A solution exists!

Existence and uniqueness of solutions
The simplest equation

Proof of uniqueness:

Suppose there are two different solutions,

$$ y_1(t), \quad y_2(t) $$

Then the function $z(t) = y_1(t) - y_2(t)$ satisfies

$$ {}_0D_t^{\sigma_n} z(t) = 0, $$

$$ \left[{}_0D_t^{\sigma_k - 1} z(t) \right]_{t=0} = 0, \qquad k = 1, 2, \dots, n $$

But then the Laplace transform of $z(t)$ is $Z(s) = 0,$ and $z(t) = 0$ almost everywhere in $(0, T)$.

Therefore, solution in $L_1(0, T)$ is unique. ■

Existence and uniqueness of solutions
General linear fractional differential equations

THEOREM 2. *If $f(t) \in L_1(0, T)$, and $p_j(t)$, $(j = 1, \ldots, n)$, are continuous functions in the closed interval $[0, T]$, then the initial-value problem*

$$ {}_0\mathcal{D}_t^{\sigma_n} y(t) + \sum_{j=1}^{n-1} p_j(t)\, {}_0\mathcal{D}_t^{\sigma_{n-j}} y(t) + p_n(t) y(t) = f(t), \qquad (0 < t < T < \infty) $$

$$ \left[{}_0\mathcal{D}_t^{\sigma_k - 1} y(t) \right]_{t=0} = b_k, \qquad k = 1, \ldots, \ n $$

has unique solution $y(t) \in L_1(0, T)$

PROOF. Let us use Theorem 1 for reducing the IVP to an integral equation of the Volterra type.

Existence and uniqueness of solutions
General linear fractional differential equations

Let us introduce an auxiliary function by denoting

$$ {}_0\mathcal{D}_t^{\sigma_n} y(t) = \varphi(t). $$

Using Theorem 1 we can write

$$ y(t) = \frac{1}{\Gamma(\sigma_n)} \int_0^t (t-\tau)^{\sigma_n - 1} \varphi(t)\, dt + \sum_{i=1}^{n} b_i \frac{t^{\sigma_i - 1}}{\Gamma(\sigma_i)}. $$

Substituting into the equation

$$ {}_0\mathcal{D}_t^{\sigma_n} y(t) + \sum_{k=1}^{n-1} p_{n-k}(t)\, {}_0\mathcal{D}_t^{\sigma_k} y(t) + p_n(t) y(t) = f(t). $$

we obtain... (see next slide)

Existence and uniqueness of solutions
General linear fractional differential equations

... (continued from the previous slide) we obtain:

$$ \varphi(t) + \int_0^t K(t,\tau)\, \varphi(\tau)\, dt = g(t), $$

where

$$ K(t,\tau) = p_n(t) \frac{(t-\tau)^{\sigma_n - 1}}{\Gamma(\sigma_n)} + \sum_{k=1}^{n-1} p_{n-k}(t) \frac{(t-\tau)^{\sigma_n - \sigma_k - 1}}{\Gamma(\sigma_n - \sigma_k)}, $$

$$ g(t) = f(t) - p_n(t) \sum_{i=1}^{n} b_i \frac{t^{\sigma_i - 1}}{\Gamma(\sigma_i)} - \sum_{k=1}^{n-1} p_{n-k}(t) \sum_{i=k+1}^{n} b_i \frac{t^{\sigma_i - \sigma_k - 1}}{\Gamma(\sigma_i - \sigma_k)}. $$

Existence and uniqueness of solutions
General linear fractional differential equations

Since the coefficients $p_j(t)$ are continuous in $[0, T]$, the kernel can be written as a weakly singular kernel

$$ K(t,\tau) = \frac{K^*(t,\tau)}{(t-\tau)^{1-\mu}}, $$

where $K^*(t,\tau)$ is continuous for $0 \le t \le T, \ \ 0 \le \tau \le T$, and

$$ \mu = \min\{\sigma_n, \sigma_n - \sigma_{n-1}, \sigma_n - \sigma_{n-2}, \ldots, \sigma_n - \sigma_1,\} = \min\{\sigma_n, \alpha_n\} $$

Similarly, $g(t) = \dfrac{g^*(t)}{t^{1-\nu}}, \qquad g^*(t)$ is continuous in $[0, T]$

$$ \nu = \min\{\sigma_1, \ldots, \sigma_n; \alpha_2, \ldots, \alpha_n\} = \min\{\alpha_1, \alpha_2, \ldots, \alpha_n\}. $$

Obviously, $0 < \mu \le 1,\ 0 < \nu \le 1.$

Existence and uniqueness of solutions
General linear fractional differential equations

It is known that the Volterra integral equation of the second kind with weakly singular kernel and with the right-hand side from $L_1(0,T)$ has a unique solution $\varphi(t) \in L_1(0,T)$. Then, according to Theorem 1, the unique solution of the problem

$$ {}_0\mathcal{D}_t^{\sigma_n} y(t) = \varphi(t). \qquad \left[{}_0\mathcal{D}_t^{\sigma_k - 1} y(t) \right]_{t=0} = b_k, \qquad k = 1, \ldots, n $$

can be determined using the formula

$$ y(t) = \frac{1}{\Gamma(\sigma_n)} \int_0^t (t-\tau)^{\sigma_n - 1} \varphi(t)\, dt + \sum_{i=1}^{n} b_i \frac{t^{\sigma_i - 1}}{\Gamma(\sigma_i)}. \quad \blacksquare $$

Existence and uniqueness of solutions
Particular case: zero initial conditions

Why?

- zero initial conditions mean the absolute beginning of the process represented by the function
- there are difficulties with numerical approximation of initial conditions of the type
- the coincidence of the Riemann-Liouville, Grunwald-Letnikov, Caputo, and Miller-Ross derivatives in the case of a proper number of zero initial conditions on the function *y(t)* and its integer-order derivatives; this coincidence withdraws misinterpretation of the problem formulation and solution

Existence and uniqueness of solutions
Particular case: zero initial conditions

Suppose $m-1 \le \sigma_n < m, \quad y^{(j)}(0) = 0, \qquad (j = 0, 1, \dots, m-1).$
Then we can replace sequential derivatives with R-L derivatives.

THEOREM 3. *If $f(t)$ and $p_j(t)$, $(j = 1, \dots, n)$ are continuous functions in the closed interval $[0, T]$, then the initial-value problem*

$$ {}_0D_t^{\sigma_n} y(t) + \sum_{j=1}^{n-1} p_j(t)\, {}_0D_t^{\sigma_{n-j}} y(t) + p_n(t) y(t) = f(t). $$

$$ y^{(j)}(0) = 0, \qquad (j = 0, 1, \dots, m-1) $$

where $m-1 \le \sigma_n < m$, $\sigma_n > \sigma_{n-1} > \sigma_{n-2} > \dots > \sigma_2 > \sigma_1 > 0$,

has a unique solution $y(t)$, which is continuous in $[0, T]$

Existence and uniqueness of solutions
General initial value problem (GIVP)

GIVP:

$$ {}_0\mathcal{D}_t^{\sigma_n} y(t) = f(t, y), $$

$$ \left[{}_0\mathcal{D}_t^{\sigma_k - 1} y(t) \right]_{t=0} = b_k, \qquad k = 1, \dots, n, $$

$$ {}_a\mathcal{D}_t^{\sigma_k} \equiv {}_aD_t^{\alpha_k}\, {}_aD_t^{\alpha_{k-1}} \dots {}_aD_t^{\alpha_1}; $$
$$ {}_a\mathcal{D}_t^{\sigma_k - 1} \equiv {}_aD_t^{\alpha_k - 1}\, {}_aD_t^{\alpha_{k-1}} \dots {}_aD_t^{\alpha_1}; $$
$$ \sigma_k = \sum_{j=1}^{k} \alpha_j, \qquad (k = 1, 2, \dots, n); $$
$$ 0 < \alpha_j \le 1, \qquad (j = 1, 2, \dots, n). $$

Let us suppose that $f(t, y)$ is defined in a domain G of a plane (t, y), and define a region $R(h, K) \subset G$ as a set of points $(t, y) \in G$, which satisfy the following inequalities:

$$ 0 < t < h, \qquad \left| t^{1-\sigma_1} y(t) - \sum_{i=1}^{n} b_i \frac{t^{\sigma_i - \sigma_1}}{\Gamma(\sigma_i)} \right| \le K $$

h and K are constant.

Existence and uniqueness of solutions
General initial value problem (GIVP)

THEOREM 3. *Let $f(t, y)$ be a real-valued continuous function, defined in the domain G, satisfying in G the Lipschitz condition with respect to y, i.e.,*

$$ |f(t, y_1) - f(t, y_2)| \le A |y_1 - y_2|, $$

such that

$$ |f(t, y)| \le M < \infty \qquad \textit{for all} \qquad (t, y) \in G. $$

Let also

$$ K \ge \frac{M h^{\sigma_n - \sigma_1 + 1}}{\Gamma(1 + \sigma_n)}. $$

Then there exists in the region $R(h, K)$ a unique and continuous solution $y(t)$ of the problem GIVP.

Existence and uniqueness of solutions
General initial value problem (GIVP): outline of proof

STEP 1.
Reduce GIVP to an equivalent fractional integral equation: performing subsequently fractional integration of order $\alpha_n, \alpha_{n-1}, \dots, \alpha_1$ with the help of the composition rule

$$ {}_a\mathbf{D}_t^{-p} \left({}_a\mathbf{D}_t^{p} f(t) \right) = f(t) - \sum_{j=1}^{k} \left[{}_a\mathbf{D}_t^{p-j} f(t) \right]_{t=a} \frac{(t-a)^{p-j}}{\Gamma(p-j+1)} $$

we obtain the equivalent integral equation:

$$ y(t) = \sum_{i=1}^{n} \frac{b_i}{\Gamma(\sigma_i)} t^{\sigma_i - 1} + \frac{1}{\Gamma(\sigma_n)} \int_0^t (t-\tau)^{\sigma_n - 1} f(\tau, y(\tau))\, d\tau. $$

Existence and uniqueness of solutions
General initial value problem (GIVP): outline of proof

STEP 2.
Define the sequence of functions $y_0(t), y_1(t), y_2(t), \dots,$

$$ y_0(t) = \sum_{i=1}^{n} \frac{b_i}{\Gamma(\sigma_i)} t^{\sigma_i - 1}, $$

$$ y_m(t) = \sum_{i=1}^{n} \frac{b_i}{\Gamma(\sigma_i)} t^{\sigma_i - 1} + \frac{1}{\Gamma(\sigma_n)} \int_0^t (t-\tau)^{\sigma_n - 1} f(\tau, y_{m-1}(\tau))\, d\tau, $$

$$ m = 1, 2, 3, \dots $$

STEP 3.
Show that the limit $\lim_{m \to \infty} y_m(t)$ exists and gives the required solution of the integral equation, which is equivalent to the considered GIVP. ■

Existence and uniqueness theorem as a method of solution

Consider the following IVP:

$$ {}_0\mathcal{D}_t^{\sigma_n} y(t) = \lambda y(t) $$

$$ \left[{}_0\mathcal{D}_t^{\sigma_k - 1} y(t) \right]_{t=0} = b_k, \qquad k = 1, \dots, n. $$

Here we have $f(t, y) = \lambda y$. Following the proof of TH-4,

$$ y_0(t) = \sum_{i=1}^{n} b_i \frac{t^{\sigma_i - 1}}{\Gamma(\sigma_i)}, \qquad y_m(t) = y_0(t) + \frac{\lambda}{\Gamma(\sigma_n)} \int_0^t (t-\tau)^{\sigma_n - 1} y_{m-1}(\tau)\, d\tau $$

$$ = y_0(t) + \lambda\, {}_0D_t^{-\sigma_n} y_{m-1}(t), $$

$$ m = 1, 2, 3, \dots $$

Existence and uniqueness theorem as a method of solution

Using fractional differentiation of the power function:

$$
\begin{aligned}
y_1(t) &= y_0(t) + \lambda\, {}_0D_t^{-\sigma_n} \left\{ \sum_{i=1}^{n} b_i \frac{t^{\sigma_i - 1}}{\Gamma(\sigma_i)} \right\} \\
&= y_0(t) + \lambda \sum_{i=1}^{n} b_i \frac{t^{\sigma_n + \sigma_i - 1}}{\Gamma(\sigma_n + \sigma_i)}, \\
y_2(t) &= y_0(t) + \lambda\, {}_0D_t^{-\sigma_n} y_1(t) \\
&= y_0(t) + \lambda\, {}_0D_t^{-\sigma_n} \left\{ y_0(t) + \lambda \sum_{i=1}^{n} b_i \frac{t^{\sigma_n + \sigma_i - 1}}{\Gamma(\sigma_n + \sigma_i)} \right\} \\
&= y_0(t) + \lambda \sum_{i=1}^{n} b_i \frac{t^{\sigma_n + \sigma_i - 1}}{\Gamma(\sigma_n + \sigma_i)} + \lambda^2 \sum_{i=1}^{n} b_i \frac{t^{2\sigma_n + \sigma_i - 1}}{\Gamma(2\sigma_n + \sigma_i)} \\
&= \sum_{i=1}^{n} b_i \sum_{k=0}^{2} \frac{\lambda^k\, t^{k\sigma_n + \sigma_i - 1}}{\Gamma(k\sigma_n + \sigma_i)},
\end{aligned}
$$

Existence and uniqueness theorem as a method of solution

It can be shown by induction that

$$
y_m(t) = \sum_{i=1}^{n} b_i \sum_{k=0}^{m} \frac{\lambda^k\, t^{k\sigma_n + \sigma_i - 1}}{\Gamma(k\sigma_n + \sigma_i)}, \qquad m = 1, 2, 3, \ldots
$$

Taking $m \to \infty$ we obtain the solution:

$$
y(t) = \sum_{i=1}^{n} b_i \sum_{k=0}^{\infty} \frac{\lambda^k\, t^{k\sigma_n + \sigma_i - 1}}{\Gamma(k\sigma_n + \sigma_i)} = \sum_{i=1}^{n} b_i\, t^{\sigma_i - 1} E_{\sigma_n, \sigma_i}(\lambda t^{\sigma_n})
$$

If $n = 1$ and $\alpha_1 = 1$, then the considered problem becomes

$$
y'(t) = \lambda y(t), \qquad y(0) = b_1,
$$

and the solution that we obtained becomes the classical one:

$$
y(t) = b_1 E_{1,1}(\lambda t) = e^{\lambda t}.
$$

Dependence of a solution on initial conditions

Introduce small changes (disturbances) in initial conditions:

$$
\left[{}_0\mathcal{D}_t^{\sigma_k - 1} y(t) \right]_{t=0} = b_k + \delta_k, \qquad k = 1, \ldots, n.
$$

THEOREM 5. *Let the assumptions of the Theorem 4 hold. If $y(t)$ is a solution of GIVP, and $\tilde{y}(t)$ is a solution of the same equation satisfying the initial conditions with disturbances δ_k, then for $0 < t \le h$ the following holds:*

$$
|y(t) - \tilde{y}(t)| \le \sum_{i=1}^{n} |\delta_i|\, t^{\sigma_i - 1} E_{\sigma_n, \sigma_i}(A t_n^{\sigma})
$$

(h is the same as in Theorem 4.)

Dependence of a solution on initial conditions

It follows from Theorem 5 that for every ε between 0 and h small changes in initial conditions cause only small changes of the solution in the interval $[\varepsilon, h]$, which does not contain zero.

On the other hand, the solution may change significantly in the interval $[0, \varepsilon]$. Indeed, if the non-disturbed initial conditions are zero ($b_k = 0$, $k = 1, 2, \ldots, n$), then the non-disturbed solution *y(t)* is continuous in $[0, \varepsilon]$, and therefore bounded. However, the solution $\tilde{y}(t)$, corresponding to the disturbed initial conditions, may contain terms of the form $\delta_i\, t^{\sigma_i - 1}/\Gamma(\sigma_i)$, which for $\sigma_i < 1$ make the disturbed solution unbounded at $t = 0$.