Fractional Derivatives (continued)

Selected approaches
Interpretations
Properties

Generalized functions approach

Convolution of f(t) and g(t)

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau.$$

Cauchy's formula

$$f^{(-n)}(t) = \frac{1}{\Gamma(n)} \int_{a}^{t} (t-\tau)^{n-1} f(\tau) d\tau,$$

$$f^{(-n)}(t) = f(t) * \frac{t^{n-1}}{\Gamma(n)}$$

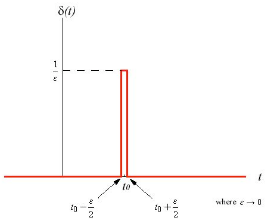
where both functions, $f(t)$ and t^{n-1}, are replaced with zeroes for $t < a$ and $t < 0$, correspondingly.

Definitions already presented...

- Grunwald-Letnikov $\quad {}_aD_t^p f(t) = \lim\limits_{\substack{h \to 0 \\ nh = t-a}} h^{-p} \sum_{r=0}^{n} (-1)^r \binom{p}{r} f(t-rh)$
- Riemann-Liouville
  - Left-sided $\quad {}_a\mathbf{D}_t^p f(t) = \frac{1}{\Gamma(k-p)} \left(\frac{d}{dt}\right)^k \int_a^t (t-\tau)^{k-p-1} f(\tau) d\tau$
  - Right-sided $\quad {}_t\mathbf{D}_b^p f(t) = \frac{1}{\Gamma(k-p)} \left(-\frac{d}{dt}\right)^k \int_t^b (\tau-t)^{k-p-1} f(\tau) d\tau$
- Caputo
  - Left-sided $\quad {}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(\alpha-n)} \int_a^t \frac{f^{(n)}(\tau) d\tau}{(t-\tau)^{\alpha+1-n}}.$
  - Right-sided $\quad$?

Generalized functions approach

Dirac's delta function $\delta(t)$



$$\int_{-\infty}^{t} \delta(t-a) dt = \begin{cases} 1, & t > a \\ 0, & t < a \end{cases} = H(t-a)$$

Heaviside's unit step function

$$H'(t-a) = \delta(t-a)$$

1. $\delta(t-a) = 0, \quad t \neq a$
2. $\int_{a-\epsilon}^{a+\epsilon} \delta(t-a) dt = 1, \quad \epsilon > 0$
3. $\int_{-\infty}^{\infty} f(t) \delta(t-a) dt = f(a)$
4. $L\{\delta(t-a); s\} = \int_0^{\infty} e^{-st} \delta(t-a) dt = e^{-as}, \quad (a > 0)$
5. $\int_{-\infty}^{\infty} f(\tau) \delta^{(k)}(t-\tau) d\tau = f^{(k)}(t)$

Generalized functions approach

Consider $\quad \Phi_p(t) = \begin{cases} \frac{t^{p-1}}{\Gamma(\gamma)}, & t > 0 \\ 0, & t \leq 0 \end{cases}$

It is known that

$$\lim_{p \to -k} \Phi_p(t) = \Phi_{-k}(t) = \delta^{(k)}(t), \qquad (k = 0, 1, 2, \ldots)$$

Integrals: $\quad f^{(-p)}(t) = f(t) * \Phi_p(t)$

Derivatives: $\quad f^{(0)}(t) = f(t) * \Phi_0(t) = f(t) * \delta(t) = f(t),$

$$f^{(1)}(t) = f(t) * \Phi_{-1}(t) = f(t) * \delta'(t) = f'(t),$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$f^{(k)}(t) = f(t) * \Phi_{-k}(t) = f(t) * \delta^{(k)}(t) = f^{(k)}(t)$$

Unification

$$\boxed{{}_a\tilde{D}_t^p f(t) = f(t) * \Phi_p(t)}$$

Generalized functions approach

Important property:

$$\Phi_p(t-a) * \Phi_q(t) = \Phi_{p+q}(t-a)$$

Indeed,

$$\begin{aligned}
\Phi_p(t-a) * \Phi_q(t) &= \int_a^t \frac{(\tau-a)^{p-1}}{\Gamma(p)} \frac{(t-\tau)^{q-1}}{\Gamma(q)} d\tau \\
&= \frac{1}{\Gamma(p)\Gamma(q)} \int_0^t (\tau-a)^{p-1}(t-\tau)^{q-1} d\tau \qquad \boxed{\tau = a + \zeta(t-a)} \\
&= \frac{(t-a)^{p+q-1}}{\Gamma(p)\Gamma(q)} \int_0^1 \zeta^{p-1}(1-\zeta)^{q-1} d\zeta \qquad \boxed{\mathrm{B}(p,q)} \\
&= \frac{(t-a)^{p+q-1}}{\Gamma(p+q)}
\end{aligned}$$

Generalized functions approach

Fractional derivative of the power function, delta, Heaviside

$$ {}_a\tilde{D}_t^p \left(\frac{(t-a)^q}{\Gamma(q+1)} \right) = \frac{(t-a)^{p-q}}{\Gamma(1+q-p)}, \quad (t>a)$$

$q = 0$

$$\boxed{ {}_a\tilde{D}_t^p H(t-a) = \frac{(t-a)^{-p}}{\Gamma(1-p)}, \quad (t>a)}$$

$q = -n-1 \ (n \geq 0)$

$$\boxed{ {}_a\tilde{D}_t^p \delta^{(n)}(t-a) = \frac{(t-a)^{-n-p-1}}{\Gamma(-n-p)}, \quad (t>a)}$$

$q - p + 1 = -n \ (n \geq 0)$

$$\boxed{ {}_a\tilde{D}_t^p \left(\frac{(t-a)^{p-n-1}}{\Gamma(p-n)} \right) = \delta^{(n)}(t-a), \quad (t>a)}$$

Generalized functions approach

Composition of fractional derivatives

If $f(t) = 0$ for $t < a$, then

$$\Big(f(t) * \Phi_p(t)\Big) * \Phi_q(t) = f(t) * \Big(\Phi_p(t) * \Phi_q(t)\Big) = f(t) * \Phi_{p+q}(t)$$

$$ {}_a\tilde{D}_t^p \Big({}_a\tilde{D}_t^q f(t) \Big) = {}_a\tilde{D}_t^q \Big({}_a\tilde{D}_t^p f(t) \Big) = {}_a\tilde{D}_t^{p+q} f(t)$$

for any p and q. This is a big advantage!

Generalized functions approach

Links between Riemann-Liouville, Grunwald-Letnikov, and Caputo fractional derivatives

Using

$$\Phi_p(t) = \begin{cases} \frac{t^{p-1}}{\Gamma(\gamma)}, & t > 0 \\ 0, & t \leq 0 \end{cases}$$

the Riemann-Liouville definition can be written as:

$$ {}_a\mathbf{D}_t^p f(t) = \frac{d^n}{dt^n} \Big(f(t) * \Phi_{n-p}(t) \Big)$$

and the Caputo derivative is:

$$ {}_a^C D_t^p f(t) = \left(\frac{d^n f(t)}{dt^n} * \Phi_{n-p}(t) \right)$$

Generalized functions approach

Fractional derivative of the power function

Power function:

$$\Phi_{q+1}(t) = \frac{t_+^q}{\Gamma(q+1)} = \begin{cases} \frac{t^q}{\Gamma(q+1)}, & (t>0) \\ 0, & (t \leq 0) \end{cases}$$

Derivative:

$$\boxed{\Phi_p(t-a) * \Phi_q(t) = \Phi_{p+q}(t-a)}$$

$$ {}_a\tilde{D}_t^p \left(\frac{(t-a)^q}{\Gamma(q+1)} \right) = \frac{(t-a)^{q-p}}{\Gamma(1+q-p)}, \quad (t>a)$$

Generalized functions approach

Links between Riemann-Liouville, Grunwald-Letnikov, and Caputo fractional derivatives

Let us suppose that the function $f(t)$ is $(n-1)$-times continuously differentiable in the interval $[a,T]$ and that $f^{(n)}(t)$ is integrable in $[a,T]$. Then for every p, $(0 < p < n)$, the Riemann-Liouville derivative ${}_a\mathbf{D}_t^p f(t)$ exists and coincides with the Grünwald-Letnikov derivative ${}_aD_t^p f(t)$, and if $0 \leq m-1 \leq p < m \leq n$, then for $a < t < T$ holds

$$ {}_a\mathbf{D}_t^p f(t) = {}_aD_t^p f(t) = \sum_{j=0}^{m-1} \frac{f^{(j)}(a)(t-a)^{j-p}}{\Gamma(1+j-p)} + \frac{1}{\Gamma(m-p)} \int_a^t \frac{f^{(m)}(\tau) d\tau}{(t-\tau)^{p-m+1}}.$$

$$ {}_a\mathbf{D}_t^p f(t) = {}_a^C D_t^p f(t) + \sum_{k=0}^{n-1} \Phi_{k-p+1}(t-a) f^{(k)}(a)$$

Generalized functions approach

Links between Riemann-Liouville and Caputo derivatives and generalized / conventional derivatives

$${}_a\mathbf{D}_t^p f(t) = {}_a^C D_t^p f(t) + \sum_{k=0}^{n-1} \Phi_{k-p+1}(t-a) f^{(k)}(a)$$

Taking $p \to n$

$${}_a^L D_t^n f(t) = {}_a^C D_t^n f(t) + \sum_{k=0}^{n-1} \delta^{(n-k-1)}(t-a) f^{(k)}(a)$$

Compare this with the known relationship:

$$\tilde{f}^{(n)}(t) = f_C^{(n)}(t) + \sum_{k=0}^{n-1} \delta^{(n-k-1)}(t-a) f^{(k)}(a)$$

Sequential fractional derivatives

... or still another way:

$$\frac{d^n f(t)}{dt^n} = \underbrace{\frac{d}{dt}\frac{d}{dt}\cdots\frac{d}{dt}}_{n} f(t).$$

$$\mathcal{D}^\alpha f(t) = D^{\alpha_1} D^{\alpha_2} \dots D^{\alpha_n} f(t) \qquad \alpha = \alpha_1 + \alpha_2 + \dots + \alpha_n$$

Generalized functions approach

Links between Riemann-Liouville and Caputo derivatives and generalized / conventional derivatives

The Riemann-Liouville definition serves as a generalization of the notion of the generalized (in the sense of distributions) derivative,

while

the Caputo derivative is a generalization of differentiation in the classical sense.

Sequential fractional derivatives

Link to the Riemann-Liouville and Caputo derivatives

Riemann-Liouville derivatives:

$${}_a\mathbf{D}_t^p f(t) = \underbrace{\frac{d}{dt}\frac{d}{dt}\cdots\frac{d}{dt}}_{n} {}_aD_t^{-(n-p)} f(t), \qquad (n-1 \le p < n)$$

Caputo derivatives

$${}_a^C D_t^p f(t) = {}_a\mathbf{D}_t^{-(n-p)} \underbrace{\frac{d}{dt}\frac{d}{dt}\cdots\frac{d}{dt}}_{n} f(t), \qquad (n-1 < p \le n).$$

Cumulative order is the same, but properties differ!

Sequential fractional derivatives

The main idea of the fractional calculus:

$$\frac{d^n f(t)}{dt^n} \Rightarrow \frac{d^\alpha f(t)}{dt^\alpha}$$

There is also another way (K.S. Miller and B. Ross):

$$\frac{d^n f(t)}{dt^n} = \underbrace{\frac{d}{dt}\frac{d}{dt}\cdots\frac{d}{dt}}_{n} f(t)$$

$$D^{n\alpha} f(t) = \underbrace{D^\alpha D^\alpha \dots D^\alpha}_{n} f(t)$$

Sequential fractional derivatives

Appear more often then one expects!

Differential equations modelling processes or objects arise as a result of a substitution of one relationship involving derivatives into another one.

If the derivatives in one or both relationships are fractional derivatives, then the resulting expression equation will contain — in a general case — sequential fractional derivatives.

The Leibniz rule

Integer order derivative of a product

$$\frac{d^n}{dt^n}\big(\varphi(t)f(t)\big) = \sum_{k=0}^{n}\binom{n}{k}\varphi^{(k)}(t)f^{(n-k)}(t)$$

Let us consider

$$\Omega_n^p(t) = \sum_{k=0}^{n}\binom{p}{k}\varphi^{(k)}(t)\,{}_aD_t^{p-k}f(t)$$

R-L differentiation of an integral depending on a parameter

We know the classical formula:

$$\frac{d}{dt}\int_0^t F(t,\tau)d\tau = \int_0^t \frac{\partial F(t,\tau)}{\partial t}d\tau + F(t,t-0).$$

Main application — in the theory of Green's functions.

The Leibniz rule

It can be shown that

$$\Omega_n^p(t) = {}_aD_t^p\big(\varphi(t)f(t)\big) + R_n^p(t)$$

$$R_n^q(t) = \frac{1}{n!\,\Gamma(-q)}\int_a^t (t-\tau)^{-q-1}f(\tau)d\tau\int_\tau^t \varphi^{(n+1)}(\xi)(\tau-\xi)^n d\xi.$$

Therefore,

$$\boxed{{}_aD_t^p\big(\varphi(t)f(t)\big) = \sum_{k=0}^{n}\binom{p}{k}\varphi^{(k)}(t)\,{}_aD_t^{p-k}f(t) - R_n^p(t)}$$

R-L differentiation of an integral depending on a parameter

Consider

$$\begin{aligned}{}_0D_t^\alpha\int_0^t K(t,\tau)d\tau &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t\frac{d\eta}{(t-\eta)^\alpha}\int_0^\eta K(\eta,\tau)d\tau\\ &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t d\tau\int_\tau^t\frac{K(\eta,\tau)d\eta}{(t-\eta)^\alpha}\\ &= \frac{d}{dt}\int_0^t\widetilde{K}(t,\tau)d\tau\\ &= \int_0^t\frac{\partial}{\partial t}\widetilde{K}(t,\tau)d\tau + \lim_{\tau\to t-0}\widetilde{K}(t,\tau)\\ &= \int_0^t {}_\tau D_t^\alpha K(t,\tau)d\tau + \lim_{\tau\to t-0}{}_\tau D_t^{\alpha-1}K(t,\tau)\end{aligned}$$

$$\boxed{\widetilde{K}(t,\xi) = \frac{1}{\Gamma(1-\alpha)}\int_\xi^t\frac{K(\eta,\xi)d\eta}{(t-\eta)^\alpha}}$$

The Leibniz rule

We can have a more elegant formula... Consider:

$$R_n^p(t) = \frac{(-1)^n}{n!\,\Gamma(-p)}\int_a^t (t-\tau)^{n-p}f(\tau)d\tau\int_0^1 \varphi^{(n+1)}(\tau+\zeta(t-\tau))\zeta^n d\zeta$$

$$= \frac{(-1)^n(t-a)^{n-p+1}}{n!\,\Gamma(-p)}\int_0^1\int_0^1 F_a(t,\zeta,\eta)d\eta d\zeta.$$

$$\boxed{\begin{aligned}\xi &= \tau+\zeta(t-\tau)\\ \tau &= a+\eta(t-a)\end{aligned}}$$

$$F_a(t,\zeta,\eta) = f(a+\eta(t-a))\varphi^{(n+1)}(a+(t-a)(\zeta+\eta-\zeta\eta))$$

If $f(\tau)$ and $\varphi(\tau)$ along with all their derivatives are continuous in [a,t], then $\lim_{n\to\infty} R_n^p(t) = 0$, and:

$$\boxed{{}_aD_t^p\big(\varphi(t)f(t)\big) = \sum_{k=0}^{\infty}\binom{p}{k}\varphi^{(k)}(t)\,{}_aD_t^{p-k}f(t)}$$

R-L differentiation of an integral depending on a parameter

Therefore,

$$\boxed{{}_0D_t^\alpha\int_0^t K(t,\tau)d\tau = \int_0^t {}_\tau D_t^\alpha K(t,\tau)d\tau + \lim_{\tau\to t-0}{}_\tau D_t^{\alpha-1}K(t,\tau) \qquad (0<\alpha<1)}$$

Important particular case: $K(t,\tau) \longrightarrow K(t-\tau)f(\tau)$

$$\boxed{{}_0D_t^\alpha\int_0^t K(t-\tau)f(\tau)d\tau = \int_0^t {}_0D_\tau^\alpha K(\tau)f(t-\tau)d\tau + \lim_{\tau\to+0} f(t-\tau)\,{}_0D_\tau^{\alpha-1}K(\tau)}$$

Behavior near lower terminal

Suppose that near $t = a$ we have $f(t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(t-a)^k$

Differentiating term-by-term:

$$ {}_aD_t^p f(t) = {}_a\mathbf{D}_t^p f(t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{\Gamma(k-p+1)}(t-a)^{k-p} $$

Therefore,

$$ \boxed{{}_aD_t^p f(t) = {}_a\mathbf{D}_t^p f(t) \sim \frac{f(a)}{\Gamma(1-p)}(t-a)^{-p}, \quad (t \to a+0)} $$

$$ \lim_{t\to a+0} {}_aD_t^p f(t) = \lim_{t\to a+0} {}_a\mathbf{D}_t^p f(t) = \begin{cases} 0, & (p<0) \\ f(a), & (p=0) \\ \infty, & (p>0) \end{cases} $$

Behavior near lower terminal

Now suppose $f(t) = (t-a)^q f_*(t)$, where $f_*(a) \neq 0$, $q > -1$, and $f_*(t)$ can be represented by the Taylor series. Then we have:

$$ {}_a\mathbf{D}_t^p f(t) = \sum_{k=0}^{\infty} \frac{f_*^{(k)}(a)}{k!} \frac{\Gamma(q+k+1)}{\Gamma(q+k-p+1)}(t-a)^{q+k-p} $$

$$ {}_a\mathbf{D}_t^p f(t) \approx \frac{f_*(a)\Gamma(q+1)}{\Gamma(q-p+1)}(t-a)^{q-p}, \quad (t \to a+0) $$

$$ \lim_{t\to a+0} {}_a\mathbf{D}_t^p f(t) = \begin{cases} 0, & (p<q) \\ \frac{f_*(a)\Gamma(q+1)}{\Gamma(q-p+1)}, & (p=q) \\ \infty, & (p>q) \end{cases} $$

Behavior far from the lower terminal

Recall the Leibniz rule:

$$ {}_aD_t^p\big(\varphi(t)f(t)\big) = \sum_{k=0}^{\infty} \binom{p}{k} \varphi^{(k)}(t)\, {}_aD_t^{p-k} f(t) $$

Take $f(t) = H(t-a)$ (the Heaviside function)

$$ \begin{aligned} {}_aD_t^p \varphi(t) &= \sum_{k=0}^{\infty} \binom{p}{k} \varphi^{(k)}(t)\, {}_aD_t^{p-k} H(t-a) \\ &= \frac{(t-a)^{-p}}{\Gamma(1-p)}\varphi(t) + \sum_{k=1}^{\infty} \binom{p}{k} \frac{(t-a)^{k-p}}{\Gamma(k-p+1)}\varphi^{(k)}(t) \\ &= \sum_{k=0}^{\infty} \binom{p}{k} \frac{(t-a)^{k-p}}{\Gamma(k-p+1)}\varphi^{(k)}(t) \end{aligned} $$

Behavior far from the lower terminal

Then, using definition of the gamma function and the reflection formula for the gamma function,

$$ \begin{aligned} {}_aD_t^p \varphi(t) &= \sum_{k=0}^{\infty} \frac{\Gamma(p+1)}{\Gamma(k+1)\Gamma(p-k+1)} \frac{(t-a)^{k-p}}{\Gamma(k-p+1)}\varphi^{(k)}(t) \\ &= \frac{\Gamma(p+1)\sin(p\pi)}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k (t-a)^{k-p}}{(p-k)\,k!}\varphi^{(k)}(t). \end{aligned} $$

Behavior far from the lower terminal

Now suppose $|t| \gg |a|$. Then

$$ (t-a)^{k-p} = t^{k-p}\left(1-\frac{a}{t}\right)^{k-p} = t^{k-p}\left(1 - \frac{(k-p)a}{t} + O\left(\frac{a^2}{t^2}\right)\right) $$

$$ (t-a)^{k-p} \approx t^{k-p} + \frac{(p-k)at^k}{t^{p+1}}, \quad (|t| \gg |a|) $$

Substituting this into the formula from the prev. slide:

$$ \begin{aligned} {}_aD_t^p \varphi(t) \approx \frac{\Gamma(p+1)\sin(p\pi)}{\pi} \Bigg\{ & \sum_{k=0}^{\infty} \frac{(-1)^k t^{k-p}}{(p-k)\,k!}\varphi^{(k)}(t) \\ & + \frac{a}{t^{p+1}} \sum_{k=0}^{\infty} \frac{(-1)^k t^k \varphi^{(k)}(t)}{k!} \Bigg\} \end{aligned} $$

Behavior far from the lower terminal

$$ {}_aD_t^p \varphi(t) = \sum_{k=0}^{\infty} \binom{p}{k} \frac{(t-a)^{k-p}}{\Gamma(k-p+1)}\varphi^{(k)}(t) $$

$$ \begin{aligned} {}_aD_t^p \varphi(t) \approx \frac{\Gamma(p+1)\sin(p\pi)}{\pi} \Bigg\{ & \sum_{k=0}^{\infty} \frac{(-1)^k t^{k-p}}{(p-k)\,k!}\varphi^{(k)}(t) \\ & + \frac{a}{t^{p+1}} \sum_{k=0}^{\infty} \frac{(-1)^k t^k \varphi^{(k)}(t)}{k!} \Bigg\} \end{aligned} $$

Therefore,

$$ {}_aD_t^p \varphi(t) \approx {}_0D_t^p \varphi(t) + \frac{a\Gamma(p+1)\sin(p\pi)\varphi(0)}{\pi t^{p+1}}, \quad (|t| \gg |a|) $$

Note this!

For large t

$$ \boxed{{}_aD_t^p \varphi(t) \approx {}_0D_t^p \varphi(t), \quad (|t| \gg |a|).} $$

Think about interpretation.

Behavior far from the lower terminal

Another way: fix t and suppose $a \to -\infty$. Then $|a| \gg |t|$

$$(t-a)^{k-p} = a^{k-p}\left(1-\frac{t}{a}\right)^{k-p} = a^{k-p}\left(1-\frac{(k-p)t}{a}+O\left(\frac{t^2}{a^2}\right)\right)$$

$$(t-a)^{k-p} \approx a^{k-p} + \frac{(p-k)ta^k}{a^{p+1}}, \quad (|t| \gg |a|)$$

$$ {}_aD_t^p\varphi(t) \approx \frac{\Gamma(p+1)\sin(p\pi)}{\pi}\left\{\sum_{k=0}^{\infty}\frac{(-1)^k(t-(t-a))^{k-p}}{(p-k)\,k!}\varphi^{(k)}(t) + \frac{t}{a^{p+1}}\sum_{k=0}^{\infty}\frac{(-1)^k(t-(t-a))^k\varphi^{(k)}(t)}{k!}\right\}$$

Note moving lower terminal!

$$ {}_aD_t^p\varphi(t) \approx {}_{t-a}D_t^p\varphi(t) + \frac{t\Gamma(p+1)\sin(p\pi)\varphi(t-a)}{\pi a^{p+1}}$$

$$\boxed{{}_aD_t^p\varphi(t) \approx {}_{t-a}D_t^p\varphi(t) \quad (|a| \gg |t|)}$$

Laplace transform — basic facts

Laplace transform of convolution:

$$f(t)*g(t) = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t f(\tau)g(t-\tau)d\tau$$

$$\boxed{L\{f(t)*g(t);\, s\} = F(s)\,G(s)}$$

Laplace transform of integer order derivatives:

$$\boxed{L\{f^n(t);\, s\} = s^nF(s) - \sum_{k=0}^{n-1}s^{n-k-1}f^{(k)}(0) = s^nF(s) - \sum_{k=0}^{n-1}s^kf^{(n-k-1)}(0)}$$

Laplace transforms of fractional derivatives

Laplace transform of R-L integral

Riemann-Liouville integral is a convolution:

$$ {}_0\mathbf{D}_t^{-p}f(t) = {}_0D_t^{-p}f(t) = \frac{1}{\Gamma(p)}\int_0^t(t-\tau)^{p-1}f(\tau)d\tau = \frac{t^{p-1}}{\Gamma(p)} * f(t)$$

$$G(s) = L\{\frac{t^{p-1}}{\Gamma(p)};\, s\} = s^{-p}$$

Therefore,

$$\boxed{L\{{}_0\mathbf{D}_t^{-p}f(t);\, s\} = L\{{}_0D_t^{-p}f(t);\, s\} = s^{-p}F(s)}$$

Laplace transform — basic facts

If $e^{-\alpha t}|f(t)| \le M$ for all $t > T$, then

$$F(s) = L\{f(t);\, s\} = \int_0^\infty e^{-st}f(t)dt$$

$$f(t) = L^{-1}\{F(s);\, t\} = \int_{c-i\infty}^{c+i\infty}e^{st}F(s)ds, \quad c = Re(s) > c_0$$

Laplace transform of convolution:

$$f(t)*g(t) = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t f(\tau)g(t-\tau)d\tau$$

$$\boxed{L\{f(t)*g(t);\, s\} = F(s)\,G(s)}$$

Laplace transform of R-L derivative

The R-L derivative can be written as

$$ {}_0\mathbf{D}_t^pf(t) = g^{(n)}(t),$$

$$g(t) = {}_0\mathbf{D}_t^{-(n-p)}f(t)\frac{1}{\Gamma(k-p)}\int_0^t(t-\tau)^{n-p-1}f(\tau)d\tau, \quad (n-1 \le p < n).$$

Using Laplace transform gives:

$$L\{{}_0\mathbf{D}_t^pf(t);\, s\} = s^nG(s) - \sum_{k=0}^{n-1}s^kg^{(n-k-1)}(0).$$

Laplace transform of R-L derivative

From the previous slide:

$$L\{{}_0\mathbf{D}_t^p f(t);\, s\} = s^n G(s) - \sum_{k=0}^{n-1} s^k g^{(n-k-1)}(0).$$

Take into account that

$$G(s) = s^{-(n-p)} F(s)$$

$$g^{(n-k-1)}(t) = \frac{d^{n-k-1}}{dt^{n-k-1}}\, {}_0\mathbf{D}_t^{-(n-p)} f(t) = {}_0\mathbf{D}_t^{p-k-1} f(t)$$

Then we obtain: unusual initial values!

$$\boxed{L\{{}_0\mathbf{D}_t^p f(t);\, s\} = s^p F(s) - \sum_{k=0}^{n-1} s^k \Big[{}_0\mathbf{D}_t^{p-k-1} f(t)\Big]_{t=0} \qquad (n-1 \le p < n)}$$

Laplace transform of sequential derivatives

Recall the Laplace transform of R-L derivative for $0 < \alpha \le 1$

$$L\{{}_0\mathbf{D}_t^\alpha f(t);\, s\} = s^\alpha F(s) - \Big[{}_0\mathbf{D}_t^{\alpha-1} f(t)\Big]_{t=0}$$

and use it m times:

$$\begin{aligned}
L\Big\{{}_0\mathcal{D}_t^{\sigma_m} f(t);\, s\Big\} &= L\Big\{{}_0D_t^{\alpha_m}\, {}_0\mathcal{D}_t^{\sigma_{m-1}} f(t);\, s\Big\} \\
&= p^{\alpha_m} L\Big\{{}_0\mathcal{D}_t^{\sigma_{m-1}} f(t);\, s\Big\} \\
&\qquad - \Big[{}_0D_t^{\alpha_m-1}\, {}_0\mathcal{D}_t^{\sigma_{m-1}} f(t)\Big]_{t=0} \\
&= p^{\alpha_m} L\Big\{{}_0\mathcal{D}_t^{\sigma_{m-1}} f(t);\, s\Big\} - \Big[{}_0\mathcal{D}_t^{\sigma_m-1} f(t)\Big]_{t=0} \\
&= p^{\alpha_m+\alpha_{m-1}} L\Big\{{}_0\mathcal{D}_t^{\sigma_{m-2}} f(t);\, s\Big\} \\
&\qquad - p^{\alpha_m} \Big[{}_0\mathcal{D}_t^{\sigma_{m-1}-1} f(t)\Big]_{t=0} \\
&\qquad\qquad - \Big[{}_0\mathcal{D}_t^{\sigma_m-1} f(t);\Big]_{t=0} \\
&\ldots \qquad \ldots \qquad \ldots \\
&= s^{\sigma_m} F(s) - \sum_{k=0}^{m-1} s^{\sigma_m-\sigma_{m-k}} \Big[{}_0\mathcal{D}_t^{\sigma_{m-k}-1} f(t)\Big]_{t=0}.
\end{aligned}$$

Laplace transform of Caputo derivative

Let us write the Caputo derivative as follows:

$${}_0^C D_t^p f(t) = {}_0\mathbf{D}_t^{-(n-p)} g(t), \qquad g(t) = f^{(n)}(t), \qquad (n-1 < p \le n)$$

Using the Laplace transform of the R-L integral gives:

$$L\{{}_0^C D_t^p f(t);\, s\} = s^{-(n-p)} G(s)$$

We also know $G(s) = s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0) = s^n F(s) - \sum_{k=0}^{n-1} s^k f^{(n-k-1)}(0).$

Therefore,

$$\boxed{L\{{}_0^C D_t^p f(t)\} = s^p F(s) - \sum_{k=0}^{n-1} s^{p-k-1} f^{(k)}(0), \qquad (n-1 < p \le n)}$$

usual initial values!

Laplace transform of sequential derivatives

The final formula is:

$$\boxed{\begin{gathered}
L\Big\{{}_0\mathcal{D}_t^{\sigma_m} f(t);\, s\Big\} = s^{\sigma_m} F(s) - \sum_{k=0}^{m-1} s^{\sigma_m-\sigma_{m-k}} \Big[{}_0\mathcal{D}_t^{\sigma_{m-k}-1} f(t)\Big]_{t=0}, \\
{}_a\mathcal{D}_t^{\sigma_{m-k}-1} \equiv {}_aD_t^{\alpha_{m-k}-1}\, {}_aD_t^{\alpha_{m-k-1}} \ldots {}_aD_t^{\alpha_1}, \\
(k = 0, 1, \ldots, m-1).
\end{gathered}}$$

1. $f(t)$ m-times differentiable, $\alpha_m = \mu,\ \alpha_k = 1,\ (k = 1, 2, \ldots, m-1)$ $\Rightarrow$ Caputo
2. $\alpha_1 = \mu,\ \alpha_k = 1,\ (k = 2, 3, \ldots, m)$ $\Rightarrow$ Riemann-Liouville
3. all $\alpha_k = 1$ $\Rightarrow$ integer-order classics

Laplace transform of sequential derivatives

Let us introduce the following notation:

$${}_a\mathcal{D}_t^{\sigma_m} \equiv {}_aD_t^{\alpha_m}\, {}_aD_t^{\alpha_{m-1}} \ldots {}_aD_t^{\alpha_1};$$

$${}_a\mathcal{D}_t^{\sigma_m-1} \equiv {}_aD_t^{\alpha_m-1}\, {}_aD_t^{\alpha_{m-1}} \ldots {}_aD_t^{\alpha_1};$$

$$\sigma_m = \sum_{j=1}^{m} \alpha_j, \qquad 0 < \alpha_j \le 1, \quad (j = 1, 2, \ldots, m)$$

Fourier transform — basic facts

$$F_e\{h(t);\, \omega\} = \int_{-\infty}^{\infty} e^{i\omega t} h(t)\, dt, \qquad h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H_e(\omega) e^{-i\omega t}\, d\omega.$$

Fourier transform of a convolution:

$$h(t) * g(t) = \int_{-\infty}^{\infty} h(t-\tau) g(\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau) g(t-\tau)\, d\tau$$

$$F_e\{h(t) * g(t);\, \omega\} = H_e(\omega)\, G_e(\omega)$$

Fourier transform of integer order derivatives:

if $h(t),\ h'(t),\ \ldots,\ h^{(n-1)}(t)$ vanish for $t \to \pm\infty$, then

$$F_e\{h^{(n)}(t);\, \omega\} = (-i\omega)^n H_e(\omega)$$

Fourier transform of the R-L integral

$$h_+(t) = \begin{cases} \frac{t^{\alpha-1}}{\Gamma(\alpha)}, & (t > 0) \\ 0, & (t \le 0) \end{cases}$$

$$F_e\{h_+(t);\, \omega\} = (-i\omega)^{-\alpha}$$

$$_{-\infty}\mathbf{D}_t^{-\alpha} f(t) = h_+(t) * g(t)$$

Using the transform of a convolution gives:

$$\boxed{F_e\left\{_{-\infty}\mathbf{D}_t^{-\alpha} g(t);\, \omega\right\} = (-i\omega)^{-\alpha}\, G(\omega)}$$

Fourier transform of fractional derivatives

If $a = -\infty$, then all definitions are identical:

$$\left.\begin{matrix} _{-\infty}\mathbf{D}_t^{\alpha} g(t) \\ _{-\infty}D_t^{\alpha} g(t) \\ _{-\infty}^{\;C}D_t^{\alpha} g(t) \end{matrix}\right\} = \frac{1}{\Gamma(n-\alpha)} \int_{-\infty}^{t} \frac{g^{(n)}(\tau)\, d\tau}{(t-\tau)^{\alpha+1-n}} = {}_{-\infty}\mathbf{D}_t^{\alpha-n} g^{(n)}(t)$$

$$(n-1 < \alpha < n)$$

Then we obtain the same expression for F-transforms:

$$\begin{aligned} F_e\{D^{\alpha} g(t);\, \omega\} &= (-i\omega)^{\alpha-n} F_e\left\{g^{(n)}(t);\, \omega\right\} \\ &= (-i\omega)^{\alpha-n}(-i\omega)^{n} G(\omega) \\ &= (-i\omega)^{\alpha} G(\omega), \end{aligned}$$

$$\boxed{F_e\{D^{\alpha} g(t);\, \omega\} = (-i\omega)^{\alpha} G(\omega)}$$