Fractional Derivatives

Selected approaches
Interpretations
Properties

The problem

$$f, \quad \frac{df}{dt}, \quad \frac{d^2 f}{dt^2}, \quad \frac{d^3 f}{dt^3}, \quad \ldots$$

$$f, \quad \int f(t)dt, \quad \int dt \int f(t)dt, \quad \int dt \int dt \int f(t)dt, \quad \ldots$$

$$\ldots, \quad \frac{d^{-2} f}{dt^{-2}}, \quad \frac{d^{-1} f}{dt^{-1}}, \quad f, \quad \frac{df}{dt}, \quad \frac{d^2 f}{dt^2}, \quad \ldots$$

How to generalize?
Which notation to use?

Which properties to preserve?

1. For integer orders must give classical derivatives/integrals
2. Zero order derivative: $\dfrac{d^0 f(t)}{dt^0} = f(t)$
3. The index law:

$$\frac{d^n}{dt^n}\frac{d^m f(t)}{dt^m} = \frac{d^m}{dt^m}\frac{d^n f(t)}{dt^n} = \frac{d^{n+m} f(t)}{dt^{n+m}}$$

4. Linearity:

$$\frac{d^n}{dt^n}\Big(\lambda f(t) + \mu g(t)\Big) = \lambda \frac{d^n f(t)}{dt^n} + \mu \frac{d^n g(t)}{dt^n}$$

Notation

$$f^{(\alpha)}(t), \quad \frac{d^\alpha f(t)}{dt^\alpha}$$

Various authors, including M. Caputo
M. Caputo,
Elasticita e Dissipazione.
Zanichelli, Bologna, 1969

$$\frac{d^\alpha f(t)}{[d(t-a)]^\alpha}, \quad \frac{d^\alpha f(t)}{[d(b-t)]^\alpha}$$

M. M. Dzhrbashyan
(various works in 1950s-1960s)

K. B. Oldham and J. Spanier,
The Fractional Calculus.
Academic Press, New York, 1974

$${}_aD_t^\alpha f(t), \quad {}_tD_b^\alpha f(t)$$

terminals

H. D. Davis,
The Theory of Linear Operators.
Principia Press, Bloomington,
Indiana, 1936

Grünwald-Letnikov approach

$$f'(t) = \frac{df}{dt} = \lim_{h\to 0} \frac{f(t) - f(t-h)}{h}$$

backward differences

$$\begin{aligned} f''(t) = \frac{d^2 f}{dt^2} &= \lim_{h\to 0} \frac{f'(t) - f'(t-h)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left\{\frac{f(t)-f(t-h)}{h} - \frac{f(t-h)-f(t-2h)}{h}\right\} \\ &= \lim_{h\to 0} \frac{f(t) - 2f(t-h) + f(t-2h)}{h^2} \end{aligned}$$

$$f'''(t) = \frac{d^3 f}{dt^3} = \lim_{h\to 0} \frac{f(t) - 3f(t-h) + 3f(t-2h) - f(t-3h)}{h^3}$$

$$f^{(n)}(t) = \frac{d^n f}{dt^n} = \lim_{h\to 0} \frac{1}{h^n} \sum_{r=0}^{n} (-1)^r \binom{n}{r} f(t - rh).$$

binomial coefficients

Grünwald-Letnikov approach

$$f_h^{(p)}(t) = \frac{1}{h^p} \sum_{r=0}^{n} (-1)^k \binom{p}{r} f(t - rh)$$

p is an arbitrary integer number; n is also integer, as above.

Obviously, for $p \le n$ we have

$$\lim_{h\to 0} f_h^{(p)}(t) = f^{(p)}(t) = \frac{d^p f}{dt^p}$$

What shall we have if p is negative?

Denote $\begin{bmatrix} p \\ r \end{bmatrix} = \frac{p(p+1)\ldots(p+r-1)}{r!}$

Then $\binom{-p}{r} = \frac{-p(-p-1)\ldots(-p-r+1)}{r!} = (-1)^r \begin{bmatrix} p \\ r \end{bmatrix}$

and we can write

$$f_h^{(-p)}(t) = h^p \sum_{r=0}^{n} \begin{bmatrix} p \\ r \end{bmatrix} f(t-rh)$$

If n is fixed, then

$$\lim_{h\to 0} f_h^{(-p)}(t) = 0$$

Not so interesting...

To arrive at non-zero limit, we must have

$n \to \infty$ as $h \to 0$

Let us take $h = \frac{t-a}{n}$ and consider the limit denoted

$$\lim_{\substack{h\to 0\\ nh=t-a}} f_h^{(-p)}(t) = {}_aD_t^{-p} f(t).$$

For inspiration, we shall consider

$p = 1$

$p = 2$

$p = 3$

$$f_h^{(-p)}(t) = h^p \sum_{r=0}^{n} \begin{bmatrix} p \\ r \end{bmatrix} f(t-rh)$$

Case of $p = 1$

$$f_h^{(-1)}(t) = h \sum_{r=0}^{n} f(t-rh)$$

integral sum

$$\lim_{\substack{h\to 0\\ nh=t-a}} f_h^{(-1)}(t) = {}_aD_t^{-1} f(t) = \int_0^{t-a} f(t-z)dz = \int_a^t f(\tau)d\tau.$$

$$f_h^{(-p)}(t) = h^p \sum_{r=0}^{n} \begin{bmatrix} p \\ r \end{bmatrix} f(t-rh)$$

Case of $p = 2$. In this case we have

$$\begin{bmatrix} 2 \\ r \end{bmatrix} = \frac{2\cdot 3\cdot \ldots \cdot (2+r-1)}{r!} = r+1$$

$$f_h^{(-2)}(t) = h \sum_{r=0}^{n} (rh+h) f(t-rh)$$

Denoting $t + h = y$

$$f_h^{(-2)}(t) = h \sum_{r=1}^{n+1} (rh) f(y-rh)$$

$$\lim_{\substack{h\to 0\\ nh=t-a}} f_h^{(-2)}(t) = {}_aD_t^{-2} f(t) = \int_0^{t-a} z f(t-z)dz = \int_a^t (t-\tau) f(\tau)d\tau$$

$$f_h^{(-p)}(t) = h^p \sum_{r=0}^{n} \begin{bmatrix} p \\ r \end{bmatrix} f(t-rh)$$

Case of $p = 3$. In this case we have

$$\begin{bmatrix} 2 \\ r \end{bmatrix} = \frac{3\cdot 4\cdot \ldots \cdot (3+r-1)}{r!} = \frac{(r+1)(r+2)}{1\cdot 2}$$

$$f_h^{(-3)}(t) = \frac{h}{1\cdot 2} \sum_{r=0}^{n} (r+1)(r+2) h^2 f(t-rh)$$

$\boxed{t+h=y}$

$$= \frac{h}{1\cdot 2} \sum_{r=1}^{n+1} r(r+1) h^2 f(y-rh)$$

$\boxed{\lim_{\substack{h\to 0\\ nh=t-a}} h \int_a^t (t-\tau) f(\tau)d\tau = 0}$

$$= \frac{h}{1\cdot 2} \sum_{r=1}^{n+1} (rh)^2 f(y-rh) + \frac{h^2}{1\cdot 2} \sum_{r=1}^{n+1} rh f(y-rh)$$

$$\lim_{\substack{h\to 0\\ nh=t-a}} f_h^{(-3)}(t) = \frac{1}{2!} \int_0^{t-a} z^2 f(t-z)dz = \frac{1}{2!} \int_a^t (t-\tau)^2 f(\tau)d\tau.$$

It can be shown by mathematical induction that

$${}_aD_t^{-p} f(t) = \lim_{\substack{h\to 0\\ nh=t-a}} h^p \sum_{r=0}^{n} \begin{bmatrix} p \\ r \end{bmatrix} f(t-rh) = \frac{1}{(p-1)!} \int_a^t (t-\tau)^{p-1} f(\tau)d\tau$$

On the other hand, we have

$$\frac{d}{dt}\left({}_aD_t^{-p} f(t)\right) = \frac{1}{(p-2!)} \int_a^t (t-\tau)^{p-2} f(\tau)d\tau = {}_aD_t^{-p+1} f(t)$$

$${}_aD_t^{-p} f(t) = \int_a^t \left({}_aD_t^{-p+1} f(t)\right) dt,$$

$${}_aD_t^{-p+1} f(t) = \int^t \left({}_aD_t^{-p+2} f(t)\right) dt,$$

and therefore

$$
\begin{aligned}
{}_aD_t^{-p}f(t) &= \int_a^t dt \int_a^t \left({}_aD_t^{-p+2}f(t) \right) dt \\
&= \int_a^t dt \int_a^t dt \int_a^t \left({}_aD_t^{-p+3}f(t) \right) dt \\
&= \underbrace{\int_a^t dt \int_a^t dt \ldots \int_a^t}_{p \text{ times}} f(t)dt.
\end{aligned}
$$

We see that the derivative of integer order p and the p -folded integral of a continuous function $f(t)$ are particular cases of the expression

$$
{}_aD_t^p f(t) = \lim_{\substack{h \to 0 \\ nh = t-a}} h^{-p} \sum_{r=0}^{n} (-1)^r \binom{p}{r} f(t - rh)
$$

Grünwald-Letnikov approach

Left-sided G-L fractional derivative:

$$
{}_aD_t^p f(t) = \lim_{\substack{h \to 0 \\ nh = t-a}} h^{-p} \sum_{r=0}^{n} (-1)^r \binom{p}{r} f(t - rh)
$$

Note that this is a generalization of backward finite differences

Grünwald-Letnikov approach

Letnikov showed that if $f(t)$ has $m+1$ continuous derivatives in $[a, b]$, then (fractional integral)

$$
{}_aD_t^{-p}f(t) = \sum_{k=0}^{m} \frac{f^{(k)}(a)(t-a)^{p+k}}{\Gamma(p+k+1)} + \frac{1}{\Gamma(p+k+1)} \int_a^t (t-\tau)^{p+m} f^{(m+1)}(\tau) d\tau
$$

This formula immediately provides us with the asymptotics of ${}_aD_t^{-p}f(t)$ at $t = a$.

Grünwald-Letnikov approach

Letnikov showed that if $f(t)$ has $m+1$ continuous derivatives in $[a, b]$, and $m > p - 1$, then (fractional derivative)

$$
{}_aD_t^p f(t) = \sum_{k=0}^{m} \frac{f^{(k)}(a)(t-a)^{-p+k}}{\Gamma(-p+k+1)} + \frac{1}{\Gamma(-p+m+1)} \int_a^t (t-\tau)^{m-p} f^{(m+1)}(\tau) d\tau
$$

The smallest possible value for m is such that

$$
m < p < m+1
$$

Grünwald-Letnikov approach

G-L fractional derivative of the power function

$$
f(t) = (t-a)^{\nu}
$$

$$
{}_aD_t^p (t-a)^{\nu} = \frac{\Gamma(\nu+1)}{\Gamma(-p+\nu+1)} (t-a)^{\nu-p}
$$

Grünwald-Letnikov approach

Composition with n-th integer order derivatives

Replace m with s such that $s \geq m + n - 1$

$$
{}_aD_t^p f(t) = \sum_{k=0}^{s} \frac{f^{(k)}(a)(t-a)^{-p+k}}{\Gamma(-p+k+1)} + \frac{1}{\Gamma(-p+s+1)} \int_a^t (t-\tau)^{s-p} f^{(s+1)}(\tau) d\tau.
$$

Consider

$$
\begin{aligned}
\frac{d^n}{dt^n} \left({}_aD_t^p f(t) \right) &= \sum_{k=0}^{s} \frac{f^{(k)}(a)(t-a)^{-p-n+k}}{\Gamma(-p-n+k+1)} \\
&\quad + \frac{1}{\Gamma(-p-n+s+1)} \int_a^t (t-\tau)^{s-p-n} f^{(s+1)}(\tau) dt = {}_aD_t^{p+n} f(t)
\end{aligned}
$$

Grünwald-Letnikov approach
Composition with n-th integer order derivatives

Since $s \geq m+n-1$ is arbitrary,
let us put $s = m+n-1$. This gives:

$$\frac{d^n}{dt^n}\left({}_aD_t^p f(t)\right) = {}_aD_t^{p+n} f(t) = \sum_{k=0}^{m+n-1} \frac{f^{(k)}(a)(t-a)^{-p-n+k}}{\Gamma(-p-n+k+1)} + \frac{1}{\Gamma(m-p)} \int_a^t (t-\tau)^{m-p-1} f^{(m+n)}(\tau)d\tau$$

Grünwald-Letnikov approach
Composition with n-th integer order derivatives

Now let us consider the reversed order:

$$ {}_aD_t^p\left(\frac{d^n f(t)}{dt^n}\right) = \sum_{k=0}^{s} \frac{f^{(n+k)}(a)(t-a)^{-p+k}}{\Gamma(-p+k+1)} + \frac{1}{\Gamma(-p+s+1)} \int_a^t (t-\tau)^{s-p} f^{(n+s+1)}(\tau)d\tau$$

$${}_aD_t^p(t-a)^\nu = \frac{\Gamma(\nu+1)}{\Gamma(-p+\nu+1)}(t-a)^{\nu-p}$$

Taking $s = m-1$

$$ {}_aD_t^p\left(\frac{d^n f(t)}{dt^n}\right) = \sum_{k=0}^{m-1} \frac{f^{(n+k)}(a)(t-a)^{-p+k}}{\Gamma(-p+k+1)} + \frac{1}{\Gamma(m-p)} \int_a^t (t-\tau)^{m-p-1} f^{(m+n)}(\tau)d\tau$$

Grünwald-Letnikov approach
Composition with n-th integer order derivatives

Now compare:

$$\frac{d^n}{dt^n}\left({}_aD_t^p f(t)\right) = {}_aD_t^{p+n} f(t) = \sum_{k=0}^{m+n-1} \frac{f^{(k)}(a)(t-a)^{-p-n+k}}{\Gamma(-p-n+k+1)} + \frac{1}{\Gamma(m-p)} \int_a^t (t-\tau)^{m-p-1} f^{(m+n)}(\tau)d\tau$$

$$ {}_aD_t^p\left(\frac{d^n f(t)}{dt^n}\right) = \sum_{k=0}^{m-1} \frac{f^{(n+k)}(a)(t-a)^{-p+k}}{\Gamma(-p+k+1)} + \frac{1}{\Gamma(m-p)} \int_a^t (t-\tau)^{m-p-1} f^{(m+n)}(\tau)d\tau$$

Grünwald-Letnikov approach
Composition with n-th integer order derivatives

Conclusion:

$$\frac{d^n}{dt^n}\left({}_aD_t^p f(t)\right) = {}_aD_t^p\left(\frac{d^n f(t)}{dt^n}\right) + \sum_{k=0}^{n-1} \frac{f^{(k)}(a)(t-a)^{-p-n+k}}{\Gamma(-p-n+k+1)}.$$

$$\frac{d^n}{dt^n}\left({}_aD_t^p f(t)\right) = {}_aD_t^p\left(\frac{d^n f(t)}{dt^n}\right) = {}_aD_t^{p+n} f(t)$$

only if $f^{(k)}(a) = 0, \quad (k = 0, 1, 2, \ldots, n-1)$

Grünwald-Letnikov approach
Composition with fractional derivatives

Similarly, it can be shown that

$${}_aD_t^q\left({}_aD_t^p f(t)\right) = {}_aD_t^p\left({}_aD_t^q f(t)\right) = {}_aD_t^{p+q} f(t)$$

where $0 \leq m < p < m+1$, $0 \leq n < q < n+1$

only if $f^{(k)}(a) = 0, \quad (k = 0, 1, \ldots, r-1), \quad r = \max(n, m)$

Riemann-Liouville approach

Manipulation with G-L fractional derivatives defined as a limit of fractional order backward difference is not convenient.

Solution? The Riemann-Liouville fractional derivative:

$${}_a\mathbf{D}_t^p f(t) = \left(\frac{d}{dt}\right)^{m+1} \int_a^t (t-\tau)^{m-p} f(\tau)d\tau, \qquad (m \leq p < m+1)$$

Riemann-Liouville approach
Direct link to the G-L fractional derivative

if $f(t)$ has $m+1$ continuous derivatives in $[a,b]$, then repeated integration by parts and differentiation gives immediately:

$$
\begin{aligned}
{}_a\mathbf{D}_t^p f(t) &= \left(\frac{d}{dt}\right)^{m+1}\int_a^t (t-\tau)^{m-p} f(\tau)\,d\tau \\
&= \sum_{k=0}^{m} \frac{f^{(k)}(a)(t-a)^{-p+k}}{\Gamma(-p+k+1)} \\
&\quad + \frac{1}{\Gamma(-p+m+1)}\int_a^t (t-\tau)^{m-p} f^{(m+1)}(\tau)\,d\tau \\
&= {}_aD_t^p f(t), \qquad (m \le p < m+1).
\end{aligned}
$$

R-L

G-L

Riemann-Liouville approach
Unification of differentiation and integration

Suppose $f(\tau)$ is continuous and integrable in every finite interval (a,t). The function $f(\tau)$ may have an integrable singularity of order $r<1$ at $\tau=a$:

$$\lim_{\tau\to a}(\tau-a)^r f(\tau) = \mathrm{const}\ (\neq 0)$$

Then the integral

$$f^{(-1)}(t) = \int_a^t f(\tau)\,d\tau$$

and tends to zero as $t\to a$.

Riemann-Liouville approach
Unification of differentiation and integration

Indeed,

$$
\begin{aligned}
\lim_{t\to a} f^{(-1)}(t) &= \lim_{t\to a}\int_a^t f(\tau)\,d\tau & \boxed{\tau = a+y(t-a)} \\
&= \lim_{t\to a}(t-a)\int_0^1 f(a+y(t-a))\,dy & \boxed{\epsilon = t-a} \\
&= \lim_{\epsilon\to 0}\epsilon^{1-r}\int_0^1 (\epsilon y)^r f(a+y\epsilon)y^{-r}\,dy = 0
\end{aligned}
$$

because $r<1$.

Therefore, we can consider two-fold integral $f^{(-2)}(t)$

Riemann-Liouville approach
Unification of differentiation and integration

$$f^{(-2)}(t) = \int_a^t d\tau_1 \int_a^{\tau_1} f(\tau)\,d\tau = \int_a^t f(\tau)\,d\tau \int_\tau^t d\tau_1 = \int_a^t (t-\tau) f(\tau)\,d\tau.$$

... and the three-fold integral ...

$$f^{(-3)}(t) = \int_a^t d\tau_1 \int_a^{\tau_1} d\tau_2 \int_a^{\tau_2} f(\tau_3)\,d\tau_3 = \int_a^t d\tau_1 \int_a^{\tau_1} (\tau_1-\tau) f(\tau)\,d\tau = \frac{1}{2}\int_a^t (t-\tau)^2 f(\tau)\,d\tau$$

... and by mathematical induction we have Cauchy's formula

$$f^{(-n)}(t) = \frac{1}{\Gamma(n)}\int_a^t (t-\tau)^{n-1} f(\tau)\,d\tau$$

Riemann-Liouville approach
Unification of differentiation and integration

Suppose that $n\ge 1$ is fixed and take $k\ge 0$. Then

$$f^{(-k-n)}(t) = \frac{1}{\Gamma(n)} D^{-k}\int_a^t (t-\tau)^{n-1} f(\tau)\,d\tau$$

where symbol D^{-k} denotes k iterated integrations.

On the other hand, for $n\ge 1$ and $k\ge n$ we have:

$$f^{(k-n)}(t) = \frac{1}{\Gamma(n)} D^{k}\int_a^t (t-\tau)^{n-1} f(\tau)\,d\tau.$$

where symbol D^k denotes k iterated differentiations.

Riemann-Liouville approach
Unification of differentiation and integration

General formula, where $n\ge 1$ is fixed:

$$f^{(k-n)}(t) = \frac{1}{\Gamma(n)} D^{k}\int_a^t (t-\tau)^{n-1} f(\tau)\,d\tau.$$

$k = n-1,\ n-2,\ \ldots$	iterated integrals
$k = n$	function $f(t)$
$k = n+1,\ n+2,\ n+3,\ \ldots$	derivatives

Riemann-Liouville approach
Unification of differentiation and integration

Now let us generalize.
Keep integer k and replace n with α :

$$ {}_a\mathbf{D}_t^{k-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \frac{d^k}{dt^k} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\,d\tau, \quad (0<\alpha\le 1) $$

or, denoting $p = k-\alpha$

$$ {}_a\mathbf{D}_t^{p} f(t) = \frac{1}{\Gamma(k-p)} \frac{d^k}{dt^k} \int_a^t (t-\tau)^{k-p-1} f(\tau)\,d\tau, \quad (k-1\le p<k) $$

or

$$ {}_a\mathbf{D}_t^{p} f(t) = \frac{d^k}{dt^k}\Big({}_a\mathbf{D}_t^{-(k-p)} f(t)\Big), \quad (k-1\le p<k) $$

Riemann-Liouville approach
Backward compatibility with integer order derivatives

Take $p = k-1$:

$$ \begin{aligned} {}_a\mathbf{D}_t^{k-1} f(t) &= \frac{d^k}{dt^k}\Big({}_a\mathbf{D}_t^{-(k-(k-1))} f(t)\Big) \\ &= \frac{d^k}{dt^k}\Big({}_a\mathbf{D}_t^{-1} f(t)\Big) = f^{(k-1)}(t) \end{aligned} $$

conventional derivative

Riemann-Liouville approach
Properties

For the fractional integrals we have:

$$ \boxed{ {}_a\mathbf{D}_t^{-p}\Big({}_a\mathbf{D}_t^{-q} f(t)\Big) = {}_a\mathbf{D}_t^{-q}\Big({}_a\mathbf{D}_t^{-p} f(t)\Big) = {}_a\mathbf{D}_t^{-p-q} f(t) } $$

This is similar to

$$ \frac{d^m}{dt^m}\left(\frac{d^n f(t)}{dt^n}\right) = \frac{d^n}{dt^n}\left(\frac{d^m f(t)}{dt^m}\right) = \frac{d^{m+n} f(t)}{dt^{m+n}} $$

Riemann-Liouville approach
Properties

For $p>0$ and $t>a$ we have:

$$ \boxed{ {}_a\mathbf{D}_t^{p}\Big({}_a\mathbf{D}_t^{-p} f(t)\Big) = f(t) } $$

but

$$ \boxed{ {}_a\mathbf{D}_t^{-p}\Big({}_a\mathbf{D}_t^{p} f(t)\Big) = f(t) - \sum_{j=1}^{k}\Big[{}_a\mathbf{D}_t^{p-j} f(t)\Big]_{t=a} \frac{(t-a)^{p-j}}{\Gamma(p-j+1)} } $$

Riemann-Liouville approach
Fractional derivative of the power function

$$ f(t) = (t-a)^\nu $$

Recall the definition of the R-L derivative:

$$ {}_a\mathbf{D}_t^{p} f(t) = \frac{d^n}{dt^n}\Big({}_a\mathbf{D}_t^{-(n-p)} f(t)\Big), \qquad (n-1\le p<n) $$

Evaluate the fractional integral (note p<0):

$$ \begin{aligned} {}_aD_t^{p}(t-a)^\nu &= \frac{1}{\Gamma(-p)}(t-a)^{\nu-p}\int_0^1 \xi^\nu (1-\xi)^{-p-1}\,d\xi \\ &= \frac{1}{\Gamma(-p)} B(-p,\nu+1)(t-a)^{\nu-p} \\ &= \frac{\Gamma(\nu+1)}{\Gamma(\nu-p+1)}(t-a)^{\nu-p}, \quad (p<0,\ \nu>-1) \end{aligned} $$

Riemann-Liouville approach
Fractional derivative of the power function

$$ f(t) = (t-a)^\nu $$

From the previous formulas we have:

$$ \boxed{ {}_a\mathbf{D}_t^{p}\Big((t-a)^\nu\Big) = \frac{\Gamma(1+\nu)}{\Gamma(1+\nu-p)}(t-a)^{\nu-p} } $$

The only restriction is that $f(t) = (t-a)^\nu$

must be integrable, that is $\nu > -1$.

Riemann-Liouville approach
Composition with integer order derivatives

$$\frac{d^n}{dt^n}\Big({}_a\mathbf{D}_t^{k-\alpha}f(t)\Big) = \frac{1}{\Gamma(\alpha)}\frac{d^{n+k}}{dt^{n+k}}\int_a^t (t-\tau)^{\alpha-1}f(\tau)d\tau = {}_a\mathbf{D}_t^{n+k-\alpha}f(t).$$

$$(0 < \alpha \le 1)$$

Denoting $p = k - \alpha$

$$\frac{d^n}{dt^n}\Big({}_a\mathbf{D}_t^{p}f(t)\Big) = {}_a\mathbf{D}_t^{n+p}f(t).$$

Riemann-Liouville approach
Composition with integer order derivatives

Take into account that

$${}_a\mathbf{D}_t^{-n}f^{(n)}(t) = \frac{1}{(n-1)!}\int_a^t (t-\tau)^{n-1}f^{(n)}(\tau)d\tau = f(t) - \sum_{j=0}^{n-1}\frac{f^{(j)}(a)(t-a)^j}{\Gamma(j+1)}$$

and

$${}_a\mathbf{D}_t^{p}g(t) = {}_a\mathbf{D}_t^{p+n}\Big({}_a\mathbf{D}_t^{-n}g(t)\Big)$$

Then

$$\begin{aligned}
{}_a\mathbf{D}_t^{p}\Big(\frac{d^n f(t)}{dt^n}\Big) &= {}_a\mathbf{D}_t^{p+n}\Big({}_a\mathbf{D}_t^{-n}f^{(n)}(t)\Big) \\
&= {}_a\mathbf{D}_t^{p+n}\Big(f(t) - \sum_{j=0}^{n-1}\frac{f^{(j)}(a)(t-a)^j}{\Gamma(j+1)}\Big) \\
&= {}_a\mathbf{D}_t^{p+n}f(t) - \sum_{j=0}^{n-1}\frac{f^{(j)}(a)(t-a)^{j-p-n}}{\Gamma(1+j-p-n)}
\end{aligned}$$

Riemann-Liouville approach
Composition with integer order derivatives

Conclusion: we have

$$\frac{d^n}{dt^n}\left({}_a\mathbf{D}_t^{p}f(t)\right) = {}_a\mathbf{D}_t^{p}\left(\frac{d^n f(t)}{dt^n}\right) = {}_a\mathbf{D}_t^{p+n}f(t)$$

only if

$$f^{(k)}(a) = 0, \qquad (k = 0, 1, 2, \ldots, n-1).$$

The same as in case of Grunwald-Letnikov derivatives!

Riemann-Liouville approach
Composition with fractional derivatives

Consider first

$$\begin{aligned}
{}_a\mathbf{D}_t^{p}\Big({}_a\mathbf{D}_t^{q}f(t)\Big) &= \frac{d^m}{dt^m}\Big\{{}_a\mathbf{D}_t^{-(m-p)}\Big({}_a\mathbf{D}_t^{q}f(t)\Big)\Big\} \\
&= \frac{d^m}{dt^m}\Big\{{}_a\mathbf{D}_t^{p+q-m}f(t) \\
&\qquad - \sum_{j=1}^{n}\Big[{}_a\mathbf{D}_t^{q-j}f(t)\Big]_{t=a}\frac{(t-a)^{m-p-j}}{\Gamma(1+m-p-j)}\Big\} \\
&= {}_a\mathbf{D}_t^{p+q}f(t) - \sum_{j=1}^{n}\Big[{}_a\mathbf{D}_t^{q-j}f(t)\Big]_{t=a}\frac{(t-a)^{-p-j}}{\Gamma(1-p-j)}.
\end{aligned}$$

Riemann-Liouville approach
Composition with fractional derivatives

Now let us interchange these derivatives
(swap p and q , and also m and n):

$${}_a\mathbf{D}_t^{q}\Big({}_a\mathbf{D}_t^{p}f(t)\Big) = {}_a\mathbf{D}_t^{p+q}f(t) - \sum_{j=1}^{m}\Big[{}_a\mathbf{D}_t^{p-j}f(t)\Big]_{t=a}\frac{(t-a)^{-q-j}}{\Gamma(1-q-j)}.$$

This is different from the previous formula.
Therefore, Riemann-Liouville derivatives,
in general, do not commute.

Riemann-Liouville approach
Composition with fractional derivatives

... R-L derivatives do not commute. With one exception:

$${}_a\mathbf{D}_t^{p}\Big({}_a\mathbf{D}_t^{q}f(t)\Big) = {}_a\mathbf{D}_t^{q}\Big({}_a\mathbf{D}_t^{p}f(t)\Big) = {}_a\mathbf{D}_t^{p+q}f(t) \qquad (p \ne q)$$

if

$$\Big[{}_a\mathbf{D}_t^{p-j}f(t)\Big]_{t=a} = 0, \quad (j = 1, 2, \ldots, m),$$

and

$$\Big[{}_a\mathbf{D}_t^{q-j}f(t)\Big]_{t=a} = 0, \quad (j = 1, 2, \ldots, n).$$

Riemann-Liouville approach

Composition with fractional derivatives

The pervious restrictions on initial values of fractional derivatives are equivalent to:

$$\left[{}_a\mathbf{D}_t^{p-j} f(t)\right]_{t=a} = 0, \quad (j = 1, 2, \ldots, m),$$

$$\boxed{f^{(j)}(a) = 0, \quad (j = 0, 1, 2, \ldots, m-1)}$$

$$\left[{}_a\mathbf{D}_t^{q-j} f(t)\right]_{t=a} = 0, \quad (j = 1, 2, \ldots, n).$$

$$\boxed{f^{(j)}(a) = 0, \quad (j = 0, 1, 2, \ldots, n-1)}$$

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Riemann-Liouville approach

Composition with fractional derivatives

Conclusion: Riemann-Liouville fractional derivatives commute, that is

$${}_a\mathbf{D}_t^p\left({}_a\mathbf{D}_t^q f(t)\right) = {}_a\mathbf{D}_t^q\left({}_a\mathbf{D}_t^p f(t)\right) = {}_a\mathbf{D}_t^{p+q} f(t)$$

if $f^{(j)}(a) = 0, \quad (j = 0, 1, 2, \ldots, r-1),$

$$r = \max(n, m).$$

The same as in case of Grunwald-Letnikov derivatives.

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Caputo approach

Why it appeared

Initial conditions for fractional differential equations with Riemann-Liouville derivatives contain

$$\lim_{t \to a} {}_a\mathbf{D}_t^{\alpha-1} f(t) = b_1,$$
$$\lim_{t \to a} {}_a\mathbf{D}_t^{\alpha-2} f(t) = b_2,$$
$$\ldots,$$
$$\lim_{t \to a} {}_a\mathbf{D}_t^{\alpha-n} f(t) = b_n,$$

and / or their combinations.
Troubles with interpretations...

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Caputo approach

M. Caputo (1967)

$${}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(\alpha - n)} \int_a^t \frac{f^{(n)}(\tau) d\tau}{(t-\tau)^{\alpha+1-n}}, \quad (n-1 < \alpha < n).$$

Note the interchange of fractional integration and integer-order differentiation compared to Riemann-Liouville approach.

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Caputo approach

Backward compatibility with integer order derivatives

Assume $0 \le n-1 < \alpha < n$ and that $f(t)$ has $n+1$ continuous derivatives. Then

$$\lim_{\alpha \to n} {}_a^C D_t^\alpha f(t) = \lim_{\alpha \to n} \left(\frac{f^{(n)}(a)(t-a)^{n-\alpha}}{\Gamma(n-\alpha+1)} + \frac{1}{\Gamma(n-\alpha+1)} \int_a^t (t-\tau)^{n-\alpha} f^{(n+1)}(\tau) d\tau \right)$$

$$= f^n(a) + \int_a^t f^{(n+1)}(\tau) d\tau = f^{(n)}(t), \quad n = 1, 2, \ldots$$

conventional derivative

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Caputo approach

Really different from Riemann-Liouville derivative

Riemann-Liouville derivative of a constant A:

$${}_0D_t^\alpha A = \frac{A t^{-\alpha}}{\Gamma(1-\alpha)}$$

Caputo derivative of a constant A:

$${}_0^C D_t^\alpha A = 0$$

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Caputo approach

Really different from Riemann-Liouville derivative

Another difference - interchange of derivatives:

$$ {}_a^C D_t^\alpha \left({}_a^C D_t^m f(t) \right) = {}_a^C D_t^m \left({}_a^C D_t^\alpha f(t) \right) = {}_a^C D_t^{\alpha+m} f(t), $$

$$ f^{(s)}(0) = 0, \quad s = n, n+1, \ldots, m $$

$$ (m = 0, 1, 2, \ldots; n-1 < \alpha < n) $$

$$ {}_a\mathbf{D}_t^m \left({}_a\mathbf{D}_t^\alpha f(t) \right) = {}_a\mathbf{D}_t^\alpha \left({}_a\mathbf{D}_t^m f(t) \right) = {}_a\mathbf{D}_t^{\alpha+m} f(t), $$

$$ f^{(s)}(0) = 0, \quad s = 0, 1, 2, \ldots, m $$

$$ (m = 0, 1, 2, \ldots; n-1 < \alpha < n) $$

Caputo: no restrictions on

$$ f^{(s)}(0), \quad (s = 0, 1, \ldots, n-1). $$

Linearity of fractional derivatives

$$ D^p\Big(\lambda f(t) + \mu g(t)\Big) = \lambda\, D^p f(t) + \mu\, D^p g(t) $$

Grunwald-Letnikov derivatives:

$$ \begin{aligned} {}_aD_t^p\Big(\lambda f(t) + \mu g(t)\Big) &= \lim_{\substack{h\to 0\\ nh=t-a}} h^{-p} \sum_{r=0}^{n} (-1)^r \binom{p}{r} \Big(\lambda f(t-rh) + \mu g(t-rh)\Big) \\ &= \lambda \lim_{\substack{h\to 0\\ nh=t-a}} h^{-p} \sum_{r=0}^{n} (-1)^r \binom{p}{r} f(t-rh) \\ &\quad + \mu \lim_{\substack{h\to 0\\ nh=t-a}} h^{-p} \sum_{r=0}^{n} (-1)^r \binom{p}{r} g(t-rh) \\ &= \lambda\, {}_aD_t^p f(t) + \mu\, {}_aD_t^p g(t). \end{aligned} $$

Linearity of fractional derivatives

$$ D^p\Big(\lambda f(t) + \mu g(t)\Big) = \lambda\, D^p f(t) + \mu\, D^p g(t) $$

Riemann-Liouville derivatives:

$$ \begin{aligned} {}_a\mathbf{D}_t^p\Big(\lambda f(t) + \mu g(t)\Big) &= \frac{1}{\Gamma(k-p)} \frac{d^k}{dt^k} \int_a^t (t-\tau)^{k-p-1} \Big(\lambda f(\tau) + \mu g(\tau)\Big) d\tau \\ &= \frac{\lambda}{\Gamma(k-p)} \frac{d^k}{dt^k} \int_a^t (t-\tau)^{k-p-1} f(\tau) d\tau \\ &\quad + \frac{\mu}{\Gamma(k-p)} \frac{d^k}{dt^k} \int_a^t (t-\tau)^{k-p-1} g(\tau) d\tau \\ &= \lambda\, {}_a\mathbf{D}_t^p f(t) + \mu\, {}_a\mathbf{D}_t^p g(t). \end{aligned} $$

Left- and right-sided fractional derivatives

$ {}_a\mathbf{D}_t^p f(t) $ — Left derivative

$ {}_t\mathbf{D}_b^p f(t) $ — Right derivative

a — the "past" of $f(t)$ — t — the "future" of $f(t)$ — b

$$ {}_a\mathbf{D}_t^p f(t) = \frac{1}{\Gamma(k-p)} \left(\frac{d}{dt}\right)^k \int_a^t (t-\tau)^{k-p-1} f(\tau) d\tau $$

$$ {}_t\mathbf{D}_b^p f(t) = \frac{1}{\Gamma(k-p)} \left(-\frac{d}{dt}\right)^k \int_t^b (\tau-t)^{k-p-1} f(\tau) d\tau $$