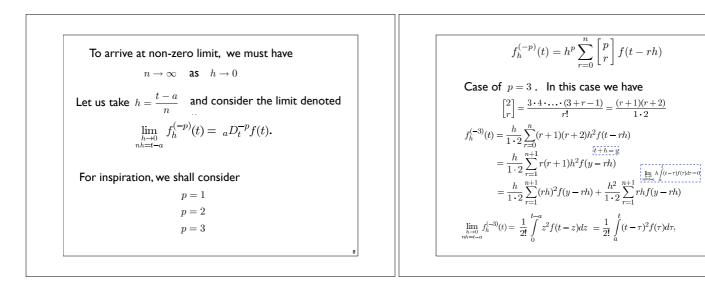


Grünwald-Letnikov approad	h
$f_h^{(p)}(t) = \frac{1}{h^p} \sum_{r=0}^n (-1)^k \binom{p}{r} f(t-rh)$	
) is an arbitrary integer number, $n$ is also integer, as	ıbove.
Diviously, for $p \le n$ we have $\lim_{h \to 0} f_h^{(p)}(t) = f^{(p)}(t) = \frac{d^p f}{dt^p}$	
What shall we have if p is negative?	

$$\begin{array}{c} \text{Denote} \quad \begin{bmatrix} p \\ r \end{bmatrix} = \frac{p(p+1)\dots(p+r-1)}{r!} \\ \text{Then} \quad \begin{pmatrix} -p \\ r \end{pmatrix} = \frac{-p(-p-1)\dots(-p-r+1)}{r!} = (-1)^r \begin{bmatrix} p \\ r \end{bmatrix} \\ \text{and we can write} \\ f_h^{(-p)}(t) = h^p \sum_{r=0}^n \begin{bmatrix} p \\ r \end{bmatrix} f(t-rh) \\ \text{fh is fixed, then} \\ \lim_{h \to 0} f_h^{(-p)}(t) = 0 \qquad \text{Not so interesting...} \end{array}$$



$$f_{h}^{(-p)}(t) = h^{p} \sum_{r=0}^{n} \begin{bmatrix} p \\ r \end{bmatrix} f(t-rh)$$
Case of  $p=1$ 

$$f_{h}^{(-1)}(t) = h \sum_{r=0}^{n} f(t-rh)$$

$$\int_{r=0}^{nherrow} f(t-rh)$$
integral sum
$$\lim_{\substack{h\to 0\\nherrow}} f_{h}^{(-1)}(t) = aD_{t}^{-1}f(t) = \int_{0}^{t-a} f(t-z)dz = \int_{a}^{t} f(\tau)d\tau.$$

$$g_{t}^{(-1)}(t) = aD_{t}^{-1}f(t) = \int_{0}^{t-a} f(t-z)dz = \int_{a}^{t} f(\tau)d\tau.$$

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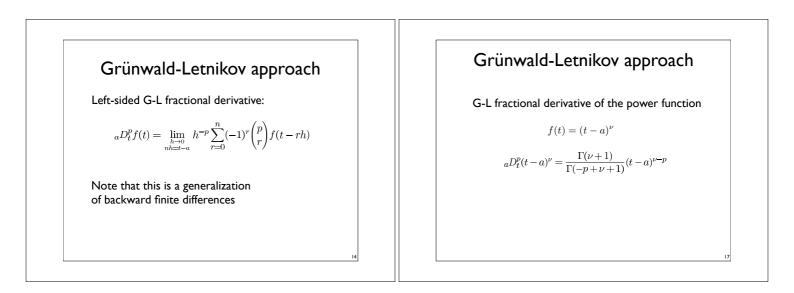
and therefore 
$${}_{a}D_{t}^{-p}f(t) = \int_{a}^{t} dt \int_{a}^{t} \left({}_{a}D_{t}^{-p+2}f(t)\right)dt$$
  
 $= \int_{a}^{t} dt \int_{a}^{t} dt \int_{a}^{t} \left({}_{a}D_{t}^{-p+3}f(t)\right)dt$   
 $= \int_{a}^{t} dt \int_{a}^{t} dt \dots \int_{a}^{t} f(t)dt.$   
We see that the derivative of integer order  $p$   
and the  $p$ -folded integral of a continuous  
function  $f(t)$  are particular cases of the  
expression  
 ${}_{a}D_{t}^{p}f(t) = \lim_{\substack{n \to 0 \\ nh \equiv t-a}} h^{-p} \sum_{r=0}^{n} (-1)^{r} {p \choose r} f(t-rh)$ 

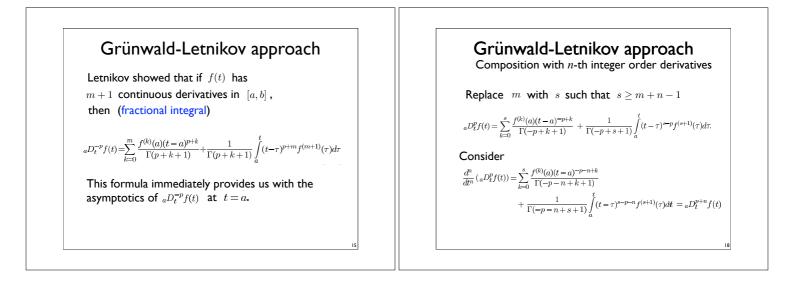
## Grünwald-Letnikov approach

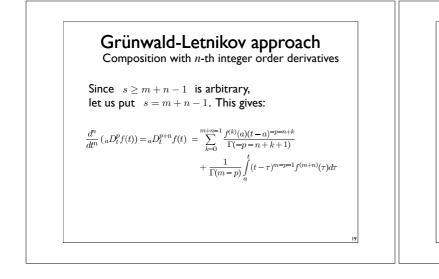
Letnikov showed that if f(t) has m+1 continuous derivatives in [a,b], and m>p-1, then (fractional derivative)

$${}_{a}D_{t}^{p}f(t) = \sum_{k=0}^{m} \frac{f^{(k)}(a)(t-a)^{-p+k}}{\Gamma(-p+k+1)} + \frac{1}{\Gamma(-p+m+1)} \int\limits_{a}^{t} (t-\tau)^{m-p} f^{(m+1)}(\tau) d\tau$$

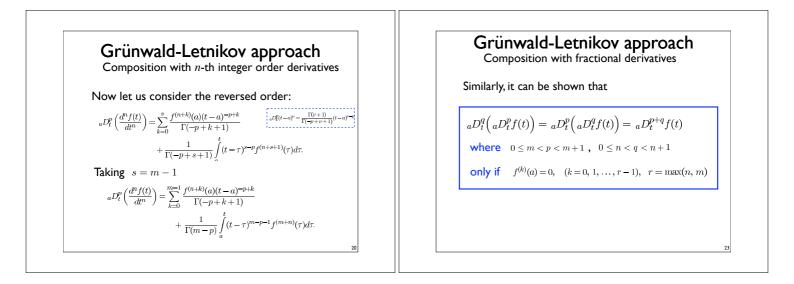
The smallest possible value for  $\,m\,$  is such that m

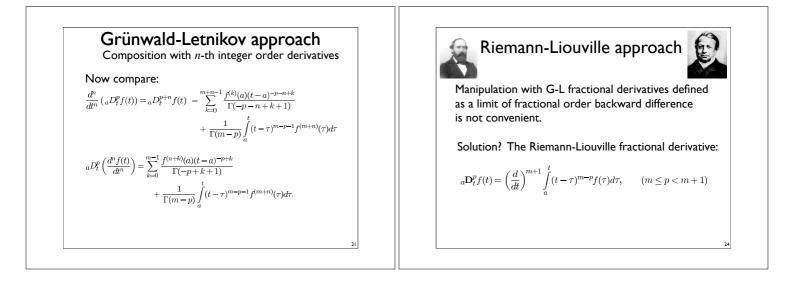


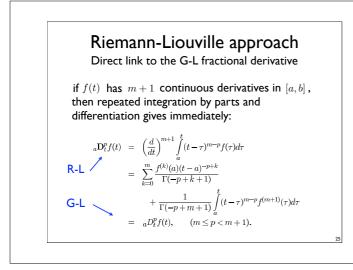


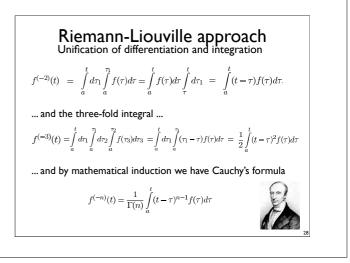


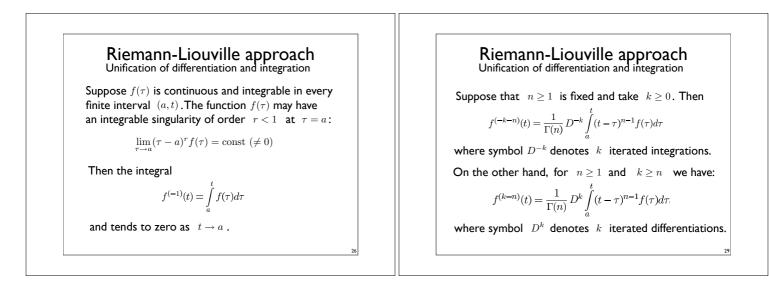
onclusion:				
$\frac{d^n}{dt^n} \left( {}_a D_t^p f \right)$	$(b)) = {}_{a}D_{t}^{p}\left(\frac{d^{n}f(t)}{dt^{n}}\right)$	$+\sum_{k=0}^{n-1} \frac{f^{(k)}(a)(t-a)}{\Gamma(-p-n+b)}$	$\frac{p-n+k}{k+1}$ .	
$\frac{d^n}{dt^n} ({}_a D_t^p$	$f(t)) = {}_{a}D_{t}^{p}\left(\frac{d}{dt}\right)$	$\frac{m}{dt^n} f(t) = a D_t^{p+n} f$ $(k = 0, 1, 2, \dots, n$	f(t)	
only if	$f^{(k)}(a) = 0,$	$(k = 0, 1, 2, \dots, n)$	-1)	

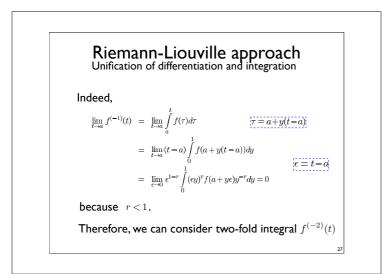


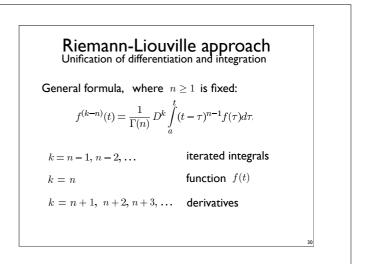


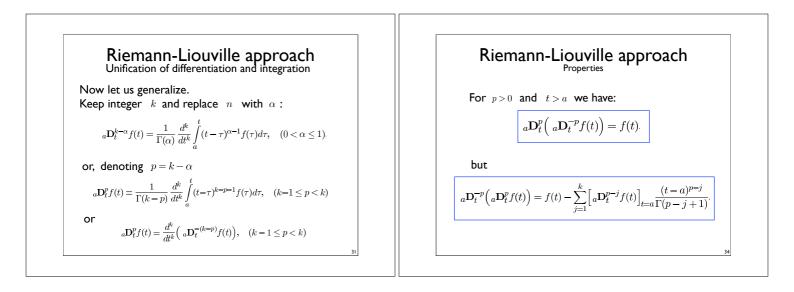


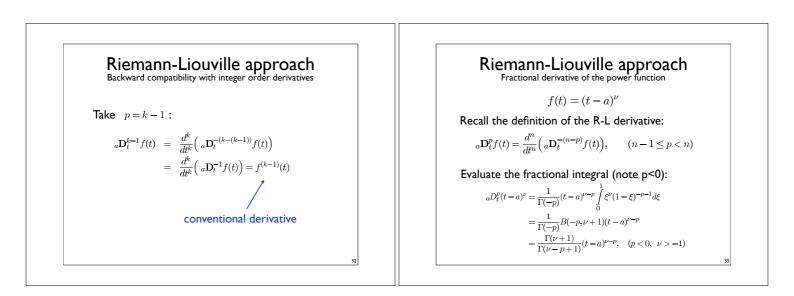


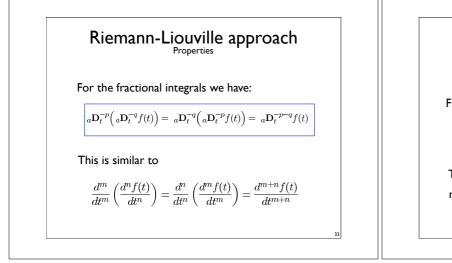




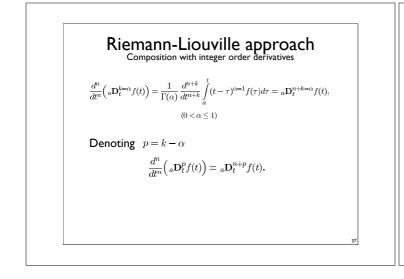


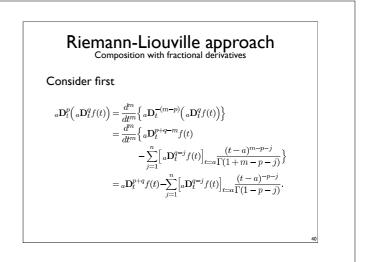


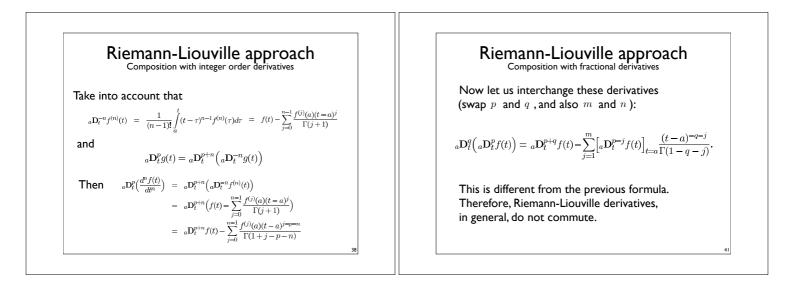




Fra	nann-Liouville approach	
	$f(t) = (t-a)^{\nu}$	
From the	previous formulas we have:	
a]	$\mathbf{D}_t^p\Big((t-a)^\nu\Big) = \frac{\Gamma(1+\nu)}{\Gamma(1+\nu-p)}(t-a)^{\nu-p}$	
The only	restriction is that $f(t) = (t-a)^{\nu}$	
must be i	ntegrable, that is $\  u>-1$ .	







Riemann-Liouville approach  
Composition with integer order derivativesConclusion: we have
$$\frac{d^n}{dt^n}(_a\mathbf{D}_t^pf(t)) = _a\mathbf{D}_t^p\left(\frac{d^nf(t)}{dt^n}\right) = _a\mathbf{D}_t^{p+n}f(t)$$
only if $f^{(k)}(a) = 0, \quad (k = 0, 1, 2, ..., n - 1).$ The same as in case of Grunwald-Letnikov derivatives!

I	Riemann-Liouville approa	ch
R-L d	erivatives do not commute.With one	exception:
$_{a}\mathbf{D}_{t}^{I}$	$P\left({}_{a}\mathbf{D}_{t}^{q}f(t)\right) = {}_{a}\mathbf{D}_{t}^{q}\left({}_{a}\mathbf{D}_{t}^{p}f(t)\right) = {}_{a}\mathbf{D}_{t}^{p+q}f(t)$	$(p\neq q)$
if		
	$\left[{}_{a}\mathbf{D}_{t}^{p-j}f(t)\right]_{t=a}=0,  (j=1,2,\ldots,m),$	
and		
	$\left[{}_{a}\mathbf{D}_{t}^{q-j}f(t)\right]_{t=a} = 0,  (j = 1, 2, \dots, n).$	

