Special Functions of the Fractional Calculus

$\Gamma(z) \qquad B(z,w)$

$E_{\alpha,\beta}(z) \qquad W(z;\alpha,\beta)$

Euler's Gamma function

Factorial: $n! = 1\cdot 2\cdot 3\cdot \ \ldots \cdot (n-1)\cdot n,$

$n! = n\cdot(n-1)!$

$0! = 1$

Leonhard Euler (1729): $\Gamma(z) = \int\limits_0^\infty e^{-t}t^{z-1}dt$

$$\Gamma(z+1) = \int\limits_0^\infty e^{-t}t^z dt = \left[-e^{-t}t^z\right]_{t=0}^{t=\infty} + z\int\limits_0^\infty e^{-t}t^{z-1}dt = z\Gamma(z)$$

consider this

$$\Gamma(x+iy) = \int\limits_0^\infty e^{-t}t^{x-1+iy}dt = \int\limits_0^\infty e^{-t}t^{x-1}e^{iy\log(t)}dt$$

$$= \int\limits_0^\infty e^{-t}t^{x-1}[\cos(y\log(t)) + i\sin(y\log(t))]dt.$$

bounded

must be $x = Re(z) > 0$

Backward compatibility with factorial

Show that $\Gamma(1) = 1$

$$\begin{aligned}
\Gamma(2) &= 1\cdot\Gamma(1) = 1 = 1!,\\
\Gamma(3) &= 2\cdot\Gamma(2) = 2\cdot 1! = 2!,\\
\Gamma(4) &= 3\cdot\Gamma(3) = 3\cdot 2! = 3!,\\
\ldots &\quad \ldots \quad \ldots \quad \ldots\\
\Gamma(n+1) &= n\cdot\Gamma(n) = n\cdot(n-1)! = n!
\end{aligned}$$

Simple poles at $z = -n, \ (n = 0, 1, 2, \ldots)$

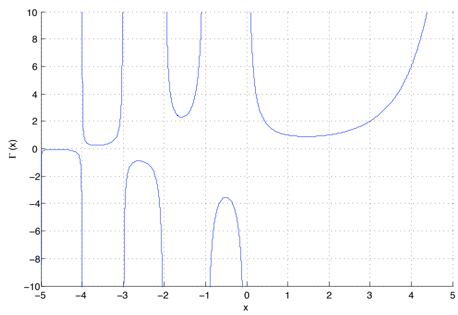
$$\Gamma(z) = \int\limits_0^1 e^{-t}t^{z-1}dt + \int\limits_1^\infty e^{-t}t^{z-1}dt.$$

$$\begin{aligned}
\int\limits_0^1 e^{-t}t^{z-1}dt &= \int\limits_0^1 \sum_{k=0}^\infty \frac{(-t)^k}{k!}t^{z-1}dt\\
&= \sum_{k=0}^\infty \frac{(-1)^k}{k!}\int\limits_0^1 t^{k+z-1}dt = \sum_{k=0}^\infty \frac{(-1)^k}{k!(k+z)}.
\end{aligned}$$

The second integral defines an *entire function* of z.

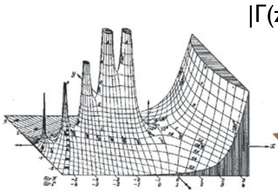
$$\Gamma(z) = \sum_{k=0}^\infty \frac{(-1)^k}{k!}\frac{1}{k+z} + \text{entire function}$$

Plot of Γ(x) for −5<x<5

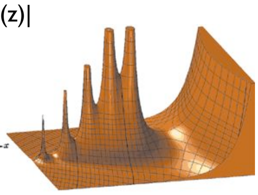


Complex gamma function

$|\Gamma(z)|$



E. Jahnke and F. Emde,
Funktionentafeln mit Formeln und Kurven.
Leipzig : B.G. Teubner, 1909



Matlab R2007a

http://www.mathworks.com/company/newsletters/news_notes/clevescorner/winter02_cleve.html

Euler's Beta function

$$B(z,w) = \int_0^1 \tau^{z-1}(1-\tau)^{w-1}d\tau, \quad (Re(z) > 0, \quad Re(w) > 0)$$

Let's establish the relationship between gamma and beta.

$$h_{z,w}(t) = \int_0^t \tau^{z-1}(1-\tau)^{w-1}d\tau \xrightarrow{\text{LT}} H_{z,w}(s) = \frac{\Gamma(z)}{s^z}\cdot\frac{\Gamma(w)}{s^w} = \frac{\Gamma(z)\Gamma(w)}{s^{z+w}}$$

Inv LT

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} \xleftarrow{t=1} h_{z,w}(t) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}t^{z+w-1}$$

$$\boxed{h_{z,w}(1) = B(z,w)}$$

Particular values of Γ(z)

can be obtained with the help of B(z,w)

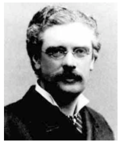
$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \xrightarrow{z=½} \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma(z)\Gamma(z+\frac{1}{2}) = \sqrt{\pi}2^{2z-1}\Gamma(2z), \quad (2z \neq 0, -1, -2, \ldots)$$

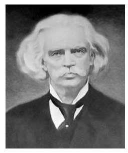
$z = n + ½$

$$\Gamma(n+\frac{1}{2}) = \frac{\sqrt{\pi}\Gamma(2n+1)}{2^{2n}\Gamma(n+1)} = \frac{\sqrt{\pi}(2n)!}{2^{2n}n!}$$

The Mittag-Leffler function



$$E_\alpha(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}$$



One of the most common - and unfounded - reasons as to why Nobel decided against a Nobel prize in math is that [a woman he proposed to/his wife/his mistress] [rejected him because of/cheated him with] a famous mathematician. Gosta Mittag-Leffler is often claimed to be the guilty party. There is no historical evidence to support the story.

In 1882 he founded the Acta Mathematica, which a century later is still one of the world's leading mathematical journals. He persuaded King Oscar II to endow prize competitions and honor various distinguished mathematicians all over Europe. Hermite, Bertrand, Weierstrass, and Poincare were among those honored by the King. (http://db.uwaterloo.ca/~alopez-o/math-faq/node50.html)

The Mittag-Leffler function

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, \qquad (\alpha > 0, \quad \beta > 0)$$

Humbert, P. and Agarwal, R.P. (1953). Sur la fonction de Mittag-Leffler et quelques-unes de ses generalisations, Bull. Sci. Math. (Ser. II) 77, 180-185.

$$E_{1,1}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty}\frac{z^k}{k!} = e^z.$$

$$E_{1,2}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+2)} = \sum_{k=0}^{\infty}\frac{z^k}{(k+1)!} = \frac{1}{z}\sum_{k=0}^{\infty}\frac{z^{k+1}}{(k+1)} = \frac{e^z-1}{z}$$

$$E_{1,3}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+3)} = \sum_{k=0}^{\infty}\frac{z^k}{(k+2)!} = \frac{1}{z^2}\sum_{k=0}^{\infty}\frac{z^{k+2}}{(k+2)} = \frac{e^z-1-z}{z^2}$$

and in general $\quad E_{1,m}(z) = \frac{1}{z^{m-1}}\left\{e^z - \sum_{k=0}^{m-2}\frac{z^k}{k!}\right\}$

The Mittag-Leffler function

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, \qquad (\alpha > 0, \quad \beta > 0)$$

$$E_{2,1}(z^2) = \sum_{k=0}^{\infty}\frac{z^{2k}}{\Gamma(2k+1)} = \sum_{k=0}^{\infty}\frac{z^{2k}}{(2k)!} = \cosh(z)$$

$$E_{2,2}(z^2) = \sum_{k=0}^{\infty}\frac{z^{2k}}{\Gamma(2k+2)} = \frac{1}{z}\sum_{k=0}^{\infty}\frac{z^{2k+1}}{(2k+1)!} = \frac{\sinh(z)}{z}$$

$$E_{\alpha,1}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)} \equiv E_\alpha(z)$$

$$E_{1/2,1}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\frac{k}{2}+1)} = e^{z^2}\text{erfc}(-z), \qquad \text{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^{\infty}e^{-t^2}dt$$

The Mittag-Leffler function

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad (\alpha > 0, \quad \beta > 0)$$

Miller-Ross function:

$$\mathcal{E}_t(\nu, a) = t^{\nu} \sum_{k=0}^{\infty} \frac{(at)^k}{\Gamma(\nu + k + 1)} = t^{\nu} E_{1,\nu+1}(at)$$

Rabotnov function:

$$Э_{\cdot}(\beta, t) = t^{\alpha} \sum_{k=0}^{\infty} \frac{\beta^k t^{k(\alpha+1)}}{\Gamma((k+1)(1+\alpha))} = t^{\alpha} E_{\alpha+1,\alpha+1}(\beta t^{\alpha+1})$$

Laplace transform of $t^k e^{\pm at}$

Consider

$$\int_0^{\infty} e^{-t} e^{zt} dt = \frac{1}{1-z} = \sum_{k=0}^{\infty} \frac{(\pm z)^k}{k!} \int_0^{\infty} e^{-t} t^k dt = \sum_{k=0}^{\infty} (\pm z)^k = \frac{1}{1 \mp z}$$

Therefore, $$\int_0^{\infty} e^{-t} e^{\pm zt} dt = \frac{1}{1 \mp z}, \quad |z| < 1$$

Differentiate k times with respect to z:

$$\int_0^{\infty} e^{-t} t^k e^{\pm zt} dt = \frac{k!}{(1-z)^{k+1}}, \quad (|z| < 1)$$

Laplace transform of $t^k e^{\pm at}$

$$\int_0^{\infty} e^{-t} t^k e^{\pm zt} dt = \frac{k!}{(1-z)^{k+1}}, \quad (|z| < 1)$$

After substitutions (which?):

$$\int_0^{\infty} e^{-pt} t^k e^{\pm at} dt = \frac{k!}{(p \mp a)^{k+1}}, \quad (Re(p) > |a|)$$

$$t^k e^{\pm at} \quad \leftrightarrows \quad \frac{k!}{(p \mp a)^{k+1}}$$

This is a well known Laplace pair.

Laplace transform of MLF

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}$$

$$\int_0^{\infty} e^{-t} t^{\beta-1} E_{\alpha,\beta}(z t^{\alpha}) dt = \frac{1}{1-z}, \qquad (|z| < 1)$$

After substitutions (which?):

$$\int_0^{\infty} e^{-pt} t^{\alpha k + \beta - 1} E^{(k)}_{\alpha,\beta}(\pm a t^{\alpha}) dt = \frac{k!\, p^{\alpha-\beta}}{(p^{\alpha} \mp a)^{k+1}}, \qquad (Re(p) > |a|^{1/\alpha}).$$

$$\boxed{t^{\alpha k + \beta - 1} E^{(k)}_{\alpha,\beta}(\pm z t^{\alpha}) \quad \leftrightarrows \quad \frac{k!\, p^{\alpha-\beta}}{(p^{\alpha} \mp a)^{k+1}}}$$

Laplace transform of MLF

Important particular case: α = β = ½

$$\int_0^{\infty} e^{-pt} t^{\frac{k-1}{2}} E^{(k)}_{\frac{1}{2},\frac{1}{2}}(\pm a\sqrt{t}) dt = \frac{k!}{(\sqrt{p} \mp a)^{k+1}}, \qquad (Re(p) > a^2).$$

Useful for solving so called semidifferential equations:

$$[D^{n/2} + A_1 D^{(n-1)/2} + \cdots + A_n D^0] y(t) = f(t)$$

Derivatives of MLF

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}$$

$${}_0D_t^{\gamma}\left(t^{\alpha k + \beta - 1} E^{(k)}_{\alpha,\beta}(\lambda t^{\alpha})\right) = t^{\alpha k + \beta - \gamma - 1} E^{(k)}_{\alpha,\beta-\gamma}(\lambda t^{\alpha})$$

Take $k = 0, \lambda = 1$ and integer γ

$$\left(\frac{d}{dt}\right)^m \left(t^{\beta-1} E_{\alpha,\beta}(t^{\alpha})\right) = t^{\beta-m-1} E_{\alpha,\beta-m}(t^{\alpha}).$$

Derivatives of MLF

We have: $$\left(\frac{d}{dt}\right)^m \left(t^{\beta-1}E_{\alpha,\beta}(t^\alpha)\right) = t^{\beta-m-1}E_{\alpha,\beta-m}(t^\alpha).$$

Take $\alpha = \frac{m}{n}$

$$\left(\frac{d}{dt}\right)^m \left(t^{\beta-1}E_{m/n,\beta}(t^{m/n})\right) = t^{\beta-1}E_{m/n,\beta}(t^{m/n}) + t^{\beta-1}\sum_{k=1}^{n}\frac{t^{-\frac{m}{n}k}}{\Gamma\left(\beta-\frac{m}{n}k\right)}$$

Setting $n = 1$ and using $\frac{1}{\Gamma(-\nu)} = 0 \quad (\nu = 0, 1, 2, \ldots)$,

$$\left(\frac{d}{dt}\right)^m \left(t^{\beta-1}E_{m,\beta}(t^m)\right) = t^{\beta-1}E_{m,\beta}(t^m),$$

$$(m = 1, 2, 3, \ldots; \quad \beta = 0, 1, 2, \ldots, m)$$

Differential equations for MLF

Denote: $$y_1(t) = t^{\beta-1}E_{m/n,\beta}(t^{m/n}),$$
$$y_2(t) = t^{\beta-1}E_{m,\beta}(t^m)$$

Then these functions satisfy the following equations:

$$\frac{d^m y_1(t)}{dt^m} - y_1(t) = t^{\beta-1}\sum_{k=1}^{n}\frac{t^{-\frac{m}{n}k}}{\Gamma\left(\beta-\frac{m}{n}k\right)},$$

$$(m, n = 1, 2, 3, \ldots)$$

$$\frac{d^m y_2(t)}{dt^m} - y_2(t) = 0,$$

$$(m = 1, 2, 3, \ldots; \quad \beta = 0, 1, 2, \ldots, m)$$

Summation formulas for MLF

$$\sum_{\nu=0}^{m-1} e^{i2\pi\nu k/m} = \begin{cases} m, & \text{if } k \equiv 0 \pmod{m} \\ 0, & \text{if } k \not\equiv 0 \pmod{m} \end{cases}$$

$$\sum_{\nu=0}^{m-1} E_{\alpha,\beta}(ze^{i2\pi\nu/m}) = mE_{m\alpha,\beta}(z^m), \quad (m \geq 1)$$

Replacing α with α/m and z with $z^{1/m}$

$$E_{\alpha,\beta}(z) = \frac{1}{m}\sum_{\nu=0}^{m-1} E_{\alpha/m,\beta}(z^{1/m}e^{i2\pi\nu/m}), \quad (m \geq 1)$$

Taking $m = 2$ and $z = t^2$ gives

$$E_{\alpha,\beta}(z) + E_{\alpha,\beta}(-z) = 2E_{\alpha,\beta}(z^2)$$

Integration of MLF

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}$$

Term-by-term integration gives:

$$\frac{1}{\Gamma(\nu)}\int_0^z (z-t)^{\nu-1}E_{\alpha,\beta}(\lambda t^\alpha)t^{\beta-1}dt = z^{\beta+\nu-1}E_{\alpha,\beta+\nu}(\lambda z^\alpha),$$

$$(\beta > 0, \quad \nu > 0).$$

Integration of MLF

$$E_{1,1}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty}\frac{z^k}{k!} = e^z;$$
$$E_{2,1}(z^2) = \sum_{k=0}^{\infty}\frac{z^{2k}}{\Gamma(2k+1)} = \sum_{k=0}^{\infty}\frac{z^{2k}}{(2k)!} = \cosh(z)$$
$$E_{2,2}(z^2) = \sum_{k=0}^{\infty}\frac{z^{2k}}{\Gamma(2k+2)} = \frac{1}{z}\sum_{k=0}^{\infty}\frac{z^{2k+1}}{(2k+1)!} = \frac{\sinh(z)}{z}$$

$$\frac{1}{\Gamma(\alpha)}\int_0^z (z-t)^{\alpha-1}e^{\lambda t}dt = z^\alpha E_{1,\alpha+1}(\lambda z), \quad (\alpha > 0),$$

$$\frac{1}{\Gamma(\alpha)}\int_0^z (z-t)^{\alpha-1}\cosh(\sqrt{\lambda}t)dt = z^\alpha E_{2,\alpha+1}(\lambda z^2), \quad (\alpha > 0),$$

$$\frac{1}{\Gamma(\alpha)}\int_0^z (z-t)^{\alpha-1}\frac{\sinh(\sqrt{\lambda}t)}{\sqrt{\lambda}t}dt = z^{\alpha+1}E_{2,\alpha+2}(\lambda z^2), \quad (\alpha > 0).$$

The Wright function

$$W(z;\alpha,\beta) = \sum_{k=0}^{\infty}\frac{z^k}{k!\,\Gamma(\alpha k+\beta)}$$

Particular cases:

$$W(z;0,1) = e^z$$

$$\left(\frac{z}{2}\right)^\nu W\left(\mp\frac{z^2}{4};1,\nu+1\right) = \begin{cases} J_\nu(z) \\ I_\nu(z) \end{cases}$$

The Wright function is a generalization of the exponential function and the Bessel functions.

Laplace transform of the W function

$$\begin{aligned} L\left\{W(t;\alpha,\beta);\ s\right\} &= L\left\{\sum_{k=0}^{\infty}\frac{t^k}{k!\,\Gamma(\alpha k+\beta)};\ s\right\} \\ &= \sum_{k=0}^{\infty}\frac{1}{\Gamma(\alpha k+\beta)}\cdot\frac{1}{s^{k+1}} \\ &= s^{-1}E_{\alpha,\beta}(s^{-1}). \end{aligned}$$

The Mittag-Leffler function!

The Mainardi function

$$M(z;\alpha) = \sum_{k=0}^{\infty}\frac{(-1)^k z^k}{k!\,\Gamma(-\alpha(k+1)+1)}$$

The M function is a particular case of the W function:

$$M(z;\alpha) = W(-z;-\alpha,1-\alpha)$$

$$M(z;\frac{1}{2}) = \frac{1}{\sqrt{\pi}}\exp\left(-\frac{z^2}{4}\right)$$