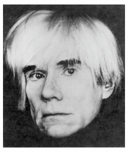
ECE 5930 003 / ECE 5930 003
Fractional order models and fractional differential equations
in science and engineering

An Overview of Fractional Order Calculus: History, Theory, Applications

Igor Podlubny
Technical University of Kosice
Slovak Republic

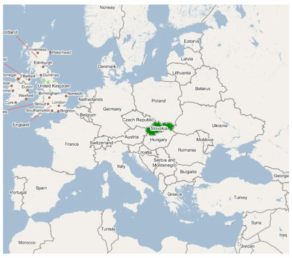
Start
1 / 90
Back
Full screen
Close
End

Do you know these Slovaks?



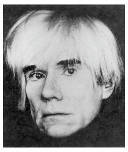

Start
4 / 90
Back
Full screen
Close
End

Slovakia



Start
2 / 90
Back
Full screen
Close
End

Do you know these Slovaks?


Andy Warhol


Eugene Cernan


John Daniel Hertz

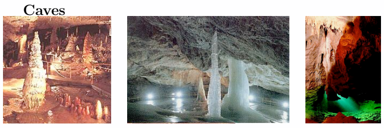

Paul Newman

Start
5 / 90
Back
Full screen
Close
End

Kosice and surroundings

Tatra Mountains

Caves

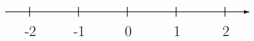


Start
3 / 90
Back
Full screen
Close
End

Contents

Start
6 / 90
Back
Full screen
Close
End

… from integer to non-integer …



$$x^n = \underbrace{x \cdot x \cdot \ldots \cdot x}_{n}$$

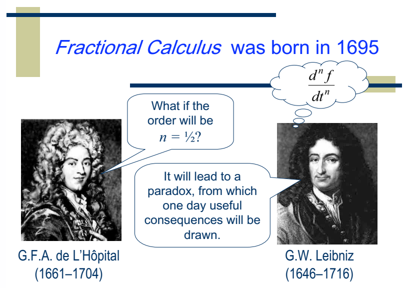
$$x^n = e^{n \ln x}$$

$$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n,$$

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \qquad x > 0,$$

$$\Gamma(n+1) = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n = n!$$

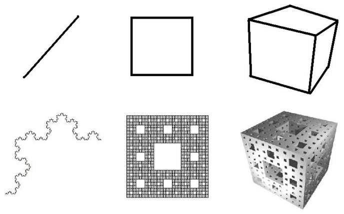
Start
7 / 90
Back
Full screen
Close
End



Start
10 / 90
Back
Full screen
Close
End

… from integer to non-integer …

$D = 1$ $\quad D = 2$ $\quad D = 3$



$D = 1.26$ $\quad D = 1.89$ $\quad D = 2.73$

Start
8 / 90
Back
Full screen
Close
End

G. W. Leibniz (1695–1697)

In the letters to J. Wallis and J. Bernulli (in 1697) Leibniz mentioned the possible approach to fractional-order differentiation in that sense, that for non-integer values of n the definition could be the following:

$$\frac{d^n e^{mx}}{dx^n} = m^n e^{mx},$$

Start
11 / 90
Back
Full screen
Close
End

Interpolation of operations

$$f, \quad \frac{df}{dt}, \quad \frac{d^2 f}{dt^2}, \quad \frac{d^3 f}{dt^3}, \quad \ldots$$

$$f, \quad \int f(t)dt, \quad \int dt \int f(t)dt, \quad \int dt \int dt \int f(t)dt, \quad \ldots$$

$$\ldots, \quad \frac{d^{-2} f}{dt^{-2}}, \quad \frac{d^{-1} f}{dt^{-1}}, \quad f, \quad \frac{df}{dt}, \quad \frac{d^2 f}{dt^2}, \quad \ldots$$

Start
9 / 90
Back
Full screen
Close
End

L. Euler (1730)

$$\frac{d^n x^m}{dx^n} = m(m-1) \ldots (m-n+1) x^{m-n}$$

$$\Gamma(m+1) = m(m-1) \ldots (m-n+1)\, \Gamma(m-n+1)$$

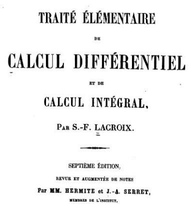
$$\frac{d^n x^m}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n}.$$

Euler suggested to use this relationship also for negative or non-integer (rational) values of n. Taking $m = 1$ and $n = \frac{1}{2}$, Euler obtained:

$$\frac{d^{1/2} x}{dx^{1/2}} = \sqrt{\frac{4x}{\pi}} \qquad \left(= \frac{2}{\sqrt{\pi}} x^{1/2} \right)$$

Start
12 / 90
Back
Full screen
Close
End

S. F. Lacroix adopted Euler's derivation for his successful textbook (*Traité du Calcul Différentiel et du Calcul Intégral*, Courcier, Paris, t. 3, 1819; pp. 409–410).


TRAITÉ ÉLÉMENTAIRE

DE

CALCUL DIFFÉRENTIEL

ET DE

CALCUL INTÉGRAL,

PAR S.-F. LACROIX.

SEPTIÈME ÉDITION,

REVUE ET AUGMENTÉE DE NOTES

Par MM. HERMITE et J.-A. SERRET,

MEMBRES DE L'INSTITUT.

J. B. J. Fourier (1820–1822)

The first step to generalization of the notion of differentiation for **arbitrary functions** was done by J. B. J. Fourier (*Théorie Analytique de la Chaleur*, Didot, Paris, 1822; pp. 499–508).

After introducing his famous formula

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(z)dz \int_{-\infty}^{\infty} \cos{(px - pz)}dp,$$

Fourier made a remark that

$$\frac{d^n f(x)}{dx^n} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(z)dz \int_{-\infty}^{\infty} \cos{(px - pz + n\frac{\pi}{2})}dp,$$

and this relationship could serve as a definition of the n-th order derivative for non-integer n.

N. H. Abel (1823–1826)

N. H. Abel: Solution de quelques problèmes à l'aide d'intégrales définies (1823). *Œuvres complètes de Niels Henrik Abel*, vol. 1, Grondahl, Christiania, 1881, pp. 11–18.

In fact, Abel solved the equation

$$\int_0^x \frac{s'(\eta)d\eta}{(x-\eta)^\alpha} = \psi(x),$$

for an arbitrary α (and not just for $\alpha = \frac{1}{2}$):

$$s(x) = \frac{\sin{(\pi\alpha)}}{\pi} x^\alpha \int_0^1 \frac{\psi(xt)dt}{(1-t)^{1-\alpha}}.$$

After that, Abel *expressed* the obtained solution with the help of an integral of order α:

$$s(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d^{-\alpha}\psi(x)}{dx^{-\alpha}}$$

J. Liouville (1832–1855)

Three approaches:

I. Following Leibniz:

$$\frac{d^m e^{ax}}{dx^n} = a^m e^{ax},$$

$$f(x) = \sum_{n=0}^{\infty} c_n\, e^{a_n x},$$

$$\frac{d^\nu f(x)}{dx^\nu} = \sum_{n=0}^{\infty} c_n\, a_n^\nu\, e^{a_n x}$$

J. Liouville (1832–1855)

Three approaches:

II. *Integrals* of non-integer order:

$$\int^\mu \Phi(x)dx^\mu = \frac{1}{(-1)^\mu\Gamma(\mu)}\int_0^\infty \Phi(x+\alpha)\alpha^{\mu-1}d\alpha$$

$$\int^\mu \Phi(x)dx^\mu = \frac{1}{\Gamma(\mu)}\int_0^\infty \Phi(x-\alpha)\alpha^{\mu-1}d\alpha$$

or (after the substitution $\tau = x + \alpha$, $\tau = x - \alpha$)

$$\int^\mu \Phi(x)dx^\mu = \frac{1}{(-1)^\mu\Gamma(\mu)}\int_x^\infty (\tau - x)^{\mu-1}\Phi(\tau)d\tau$$

$$\int^\mu \Phi(x)dx^\mu = \frac{1}{\Gamma(\mu)}\int_{-\infty}^x (x-\tau)^{\mu-1}\Phi(\tau)d\tau.$$

J. Liouville (1832–1855)

Three approaches:

III. *Derivatives* of non-integer order:

$$\begin{aligned}\frac{d^\mu F(x)}{dx^\mu} &= \frac{(-1)^\mu}{h^\mu}(F(x) - \frac{\mu}{1}F(x+h) + \\ &\quad + \frac{\mu(\mu-1)}{1\cdot 2}F(x+2h) - \dots)\end{aligned}$$

$$\begin{aligned}\frac{d^\mu F(x)}{dx^\mu} &= \frac{1}{h^\mu}(F(x) - \frac{\mu}{1}F(x-h) + \\ &\quad + \frac{\mu(\mu-1)}{1\cdot 2}F(x-2h) - \dots).\end{aligned}$$

(Equality is in the sense $\lim\limits_{h\to 0}$).

Liouville was the first, who realized the possibility of considereation of *left-sided* and *right-sided* fractional integrals and derivatives.

G. F. B. Riemann (1847; 1876)

Riemann used a generalization of the Taylor series for obtaining a formula for fractional-order integration, which is given below in contemporary notation:

$$D^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_c^x (x-t)^{\nu-1} f(t)\,dt + \psi(t)$$

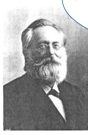
Riemann introduced an arbitrary (complementary) function $\psi(x)$ because he did not fixed the lower bound of integration c – a disadvantage, which cannot be removed in the framework of his approach.

Start 19 / 90 Back Full screen Close End

Grünwald–Letnikov definition

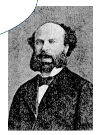
$${}_aD_t^\alpha f(t) = \lim_{h\to 0} h^{-\alpha} \sum_{j=0}^{\left[\frac{t-a}{h}\right]} (-1)^j \binom{\alpha}{j} f(t-jh)$$

$[x]$ – integer part of x



A.K. Grünwald

A.V. Letnikov



Start 22 / 90 Back Full screen Close End

N. Ya. Sonin (1869)
A. V. Letnikov (1872)
H. Laurent (1884)
N. Nekrasov (1888)
K. Nishimoto (1987–)

Cauchy's formula:

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(t)}{(t-z)^{n+1}} dt$$

For non-integer $n = \nu$ a branch point of the function $(t-z)^{-\nu-1}$ appears instead of a pole:

$$D^{\nu} f(z) = \frac{\Gamma(\nu+1)}{2\pi i} \int_c^{x^+} \frac{f(t)}{(t-z)^{\nu+1}} dt$$

Start 20 / 90 Back Full screen Close End

Some other definitions

M. Caputo (1967):

$${}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(\alpha-n)} \int_a^t \frac{f^{(n)}(\tau)\,d\tau}{(t-\tau)^{\alpha+1-n}}, \quad (n-1 \le \alpha < n)$$

K.S. Miller, B. Ross (1993):

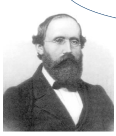
$$D^{\bar{\alpha}} f(t) = D^{\alpha_1} D^{\alpha_2} \dots D^{\alpha_n} f(t), \quad \bar{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n)$$

Start 23 / 90 Back Full screen Close End

Riemann–Liouville definition

$${}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{f(\tau)\,d\tau}{(t-\tau)^{\alpha-n+1}}$$

$$(n-1 \le \alpha < n)$$



G.F.B. Riemann (1826–1866)

J. Liouville (1809–1882)

Start 21 / 90 Back Full screen Close End

„Left " and „right" fractional derivatives

"Left - sided": $${}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{f(\tau)\,d\tau}{(t-\tau)^{\alpha-n+1}}$$

"Right - sided": $${}_tD_b^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dt}\right)^n \int_t^b \frac{f(\tau)\,d\tau}{(\tau-t)^{\alpha-n+1}}$$

${}_aD_t^\alpha f(t)$ ${}_tD_b^\alpha f(t)$

a „past" of $f(t)$ t „future" of $f(t)$ b

Start 24 / 90 Back Full screen Close End

Approximation

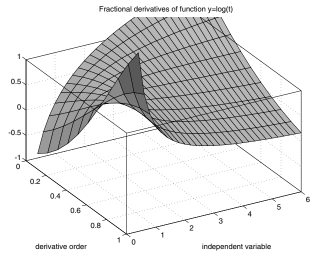
On equidistant mesh with step h the following approximation using *finite differences of fractional order,* which comes from the Grünwald–Letnikov definition:

$$ {}_aD_t^{\alpha} f(t) \approx \frac{1}{h^{\alpha}} \sum_{j=0}^{\left[\frac{t-a}{h}\right]} (-1)^j \binom{\alpha}{j} f(t-jh), $$

$[x]$- integer part of x

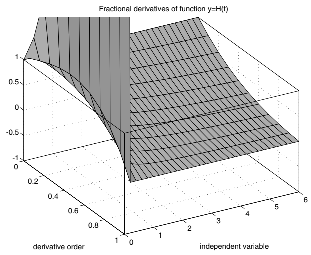
Start
Back
Full screen
Close
End
25 / 90

Example: $\ln(t)$



Start
Back
Full screen
Close
End
28 / 90

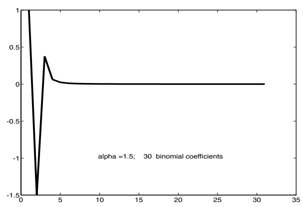
Example: Heaviside's unit step



Start
Back
Full screen
Close
End
26 / 90

"Short memory" principle

Coefficients in the Grünwald-Letnikov formula:

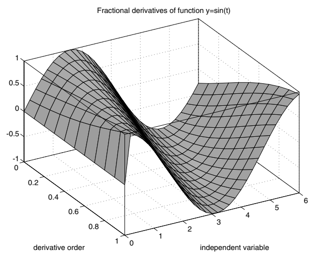


$$ {}_aD_t^{\alpha} f(t) \approx {}_{t-L}D_t^{\alpha} f(t), \qquad (t > a + L) $$

Length of "memory", L, depends on the required precision of computations.

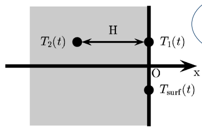
Start
Back
Full screen
Close
End
29 / 90

Example: $\sin(t)$



Start
Back
Full screen
Close
End
27 / 90

Short-memory principle in practice: heat flux through the blast furnace wall



$$ c\rho \frac{\partial u}{\partial t} = \lambda \frac{\partial^2 u}{\partial x^2} $$

J.B.J. Fourier (1768–1830)

$$ q(t) = \sqrt{c\rho\lambda}\ {}_0D_t^{1/2} f(t) \approx \sqrt{c\rho\lambda}\ {}_{t-L}D_t^{1/2} f(t), $$

$$ f(t) = T_{surf}(t) - T_0 $$

Start
Back
Full screen
Close
End
30 / 90

Fractional differential equations

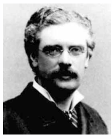
$$y^{(\alpha_n)}(t) = F(t, y^{(\alpha_1)}(t), y^{(\alpha_2)}(t), \dots, y^{(\alpha_{n-1})}(t)), \quad a<t<b$$

$$(0<\alpha_1<\alpha_2<\dots<\alpha_{n-1}<\alpha_n, \quad n-1<\alpha\le n)$$

$$y^{(k)}(a)=0, \quad k=0,1,\dots,n-1$$

Start
31 / 90
Back
Full screen
Close
End

G. M. Mittag-Leffler



Professor Donald E. Knuth, creator of TEX:

"As far as the spacing in mathematics is concerned... I took *Acta Mathematica*, from 1910 approximately; this was a journal in Sweden ... Mittag-Leffler was the editor, and his wife was very rich, and they had the highest budget for making quality mathematics printing. So the typography was especially good in *Acta Mathematica*."

(Questions and Answers with Prof. Donald E. Knuth, Charles University, Prague, March 1996)

Start
32 / 90
Back
Full screen
Close
End

Mittag-Leffler function: definition

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad (\alpha>0, \quad \beta>0)$$

$$E_{1,1}(z) = e^z,$$

$$E_{2,1}(z^2) = \cosh(z), \quad E_{2,2}(z^2) = \frac{\sinh(z)}{z}.$$

$$E_{1/2,1}(z) = e^{z^2}\operatorname{erfc}(-z);$$

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-t^2} dt.$$

Start
33 / 90
Back
Full screen
Close
End

Mittag-Leffler function: most important properties

Laplace transform of the M-L function:

$$\int_0^{\infty} e^{-st} t^{\alpha k+\beta-1} E^{(k)}_{\alpha,\beta}(\pm a t^{\alpha}) dt = \frac{k!\, s^{\alpha-\beta}}{(s^{\alpha} \mp a)^{k+1}},$$

$$(Re(s) > |a|^{1/\alpha}).$$

Differentiation of fractional order:

$${}_0D_t^{\gamma}(t^{\alpha k+\beta-1} E^{(k)}_{\alpha,\beta}(\lambda t^{\alpha})) = t^{\alpha k+\beta-\gamma-1} E^{(k)}_{\alpha,\beta-\gamma}(\lambda t^{\alpha})$$

Start
34 / 90
Back
Full screen
Close
End

Analytical methods for FDEs

- Integral transforms (Laplace, Fourier, Mellin)
- Power series method
- Babenko's symbolic method
- Method of orthogonal polynomials
- Fractional Green's function

Start
35 / 90
Back
Full screen
Close
End

Laplace transform method

LT of the Riemann-Liouville derivative:

$$L\{\, {}_0D_t^{\alpha} f(t);\ s\} = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^k \left[{}_0D_t^{\alpha-k-1} f(t)\right]_{t=0}$$

$$(n-1<\alpha\le n).$$

Example LT-1:

$${}_0D_t^{1/2} f(t) + a f(t) = 0, \quad (t>0);$$

$$\left[{}_0D_t^{-1/2} f(t)\right]_{t=0} = f_0$$

Solution LT-1:

$$F(s) = \frac{f_0}{s^{1/2}+a},$$

$$f(t) = f_0\, t^{-1/2} E_{\frac12,\frac12}(-a\sqrt{t}).$$

Start
36 / 90
Back
Full screen
Close
End

Example LT-2:

$$ {}_0D_t^{\alpha} y(t) - \lambda y(t) = h(t), \quad (t > 0); $$

$$ \left[{}_0D_t^{\alpha-k} y(t) \right]_{t=0} = b_k, \quad (k = 1, 2, \ldots, n), $$

$$ (n-1 < \alpha < n) $$

Solution LT-2:

$$ s^{\alpha} Y(s) - \lambda Y(s) = H(s) + \sum_{k=1}^{n} b_k s^{k-1} $$

$$ Y(s) = \frac{H(s)}{s^{\alpha} - \lambda} + \sum_{k=1}^{n} b_k \frac{s^{k-1}}{s^{\alpha} - \lambda} $$

$$ y(t) = \sum_{k=1}^{n} b_k t^{\alpha-k} E_{\alpha,\alpha-k+1}(\lambda t^{\alpha}) + $$

$$ + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(\lambda (t-\tau)^{\alpha}) h(\tau) d\tau $$

Fractional Green's function: definition

$$ (*) \qquad {}_0\mathcal{L}_t y(t) = f(t); \qquad {}_0\mathcal{D}_t^{\sigma_k - 1} y(t) \Big|_{t=0} = b_k, \quad k = 1, \ldots, n $$

$$ {}_a\mathcal{L}_t y(t) \equiv {}_a\mathcal{D}_t^{\sigma_n} y(t) + \sum_{k=1}^{n-1} p_k(t) \, {}_a\mathcal{D}_t^{\sigma_{n-k}} y(t) + p_n(t) y(t), $$

$$ {}_a\mathcal{D}_t^{\sigma_k} \equiv {}_aD_t^{\alpha_k} \, {}_aD_t^{\alpha_{k-1}} \ldots {}_aD_t^{\alpha_1}; \quad {}_a\mathcal{D}_t^{\sigma_k - 1} \equiv {}_aD_t^{\alpha_k - 1} \, {}_aD_t^{\alpha_{k-1}} \ldots {}_aD_t^{\alpha_1}; $$

$$ \sigma_k = \sum_{j=1}^{k} \alpha_j, \quad (k = 1, 2, \ldots, n); \qquad 0 \le \alpha_j \le 1, \quad (j = 1, 2, \ldots, n). $$

Definition. Function $G(t, \tau)$ satisfying the following conditions

a) ${}_\tau\mathcal{L}_t G(t, \tau) = 0$ for every $\tau \in (0, t)$;

b) $\lim_{\tau \to t-0} ({}_\tau\mathcal{D}_t^{\sigma_k - 1} G(t, \tau)) = \delta_{k,n}, \quad k = 0, 1, \ldots, n,$
($\delta_{k,n}$ is Kronecker's delta);

c) $\lim\limits_{\substack{\tau, t \to +0 \\ \tau < t}} ({}_\tau\mathcal{D}_t^{\sigma_k} G(t, \tau)) = 0, \quad k = 0, 1, \ldots, n-1$

is called Green's function of equation **(*)**.

Fractional Green's function: utilization

$$ y(t) = \int_0^t G(t, \tau) f(\tau) d\tau $$

For equations with constant coefficients:

$$ y(t) = \sum_{k=1}^{n} b_k \psi_k(t) + \int_0^t G(t-\tau) f(\tau) d\tau, $$

$$ b_k = {}_0\mathcal{D}_t^{\sigma_k - 1} y(t) \Big|_{t=0} $$

$$ \psi_k(t) = {}_0\mathcal{D}_t^{\sigma_n - \sigma_k} G(t), \qquad {}_0\mathcal{D}_t^{\sigma_n - \sigma_k} \equiv {}_aD_t^{\alpha_n} \, {}_aD_t^{\alpha_{n-1}} \ldots {}_aD_t^{\alpha_{k+1}} $$

Fractional Green's function: examples

One-term equation:

$$ a \, {}_0D_t^{\alpha} y(t) = f(t) $$

$$ G_1(t) = \frac{1}{a} \frac{t^{\alpha-1}}{\Gamma(\alpha)} $$

Two-term equation:

$$ a \, {}_0D_t^{\alpha} y(t) + b y(t) = f(t) $$

$$ G_2(t) = \frac{1}{a} t^{\alpha-1} E_{\alpha,\alpha}(-\frac{b}{a} t^{\alpha}) $$

Fractional Green's function: examples

Three-term equation:

$$ a \, {}_0D_t^{\beta} y(t) + b \, {}_0D_t^{\alpha} y(t) + c \, y(t) = f(t) $$

$$ G_3(t) = \frac{1}{a} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{c}{a} \right)^k \times $$

$$ \times \, t^{\beta(k+1)-1} E^{(k)}_{\beta-\alpha, \beta+\alpha k}(-\frac{b}{a} t^{\beta-\alpha}) $$

Numerical methods for FDEs

- Fractional order differences
- Matrix approach
- Quadrature formulas
- "Short memory" principle

$$ {}_aD_t^{\alpha} f(t) \approx {}_{t-L}D_t^{\alpha} f(t) $$

Matrix approach to discretization of fractional derivatives and fractional integrals

Triangular strip matrices (TSM)

Lower TSM:

$$L_N = \begin{bmatrix} \omega_0 & 0 & 0 & 0 & \cdots & 0 \\ \omega_1 & \omega_0 & 0 & 0 & \cdots & 0 \\ \omega_2 & \omega_1 & \omega_0 & 0 & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ldots & \ldots \\ \omega_{N-1} & \ddots & \omega_2 & \omega_1 & \omega_0 & 0 \\ \omega_N & \omega_{N-1} & \ddots & \omega_2 & \omega_1 & \omega_0 \end{bmatrix},$$

Upper TSM:

$$U_N = \begin{bmatrix} \omega_0 & \omega_1 & \omega_2 & \ddots & \omega_{N-1} & \omega_N \\ 0 & \omega_0 & \omega_1 & \ddots & \ddots & \omega_{N-1} \\ 0 & 0 & \omega_0 & \ddots & \omega_2 & \ddots \\ 0 & 0 & 0 & \ddots & \omega_1 & \omega_2 \\ \cdots & \cdots & \cdots & \cdots & \omega_0 & \omega_1 \\ 0 & 0 & 0 & \cdots & 0 & \omega_0 \end{bmatrix},$$

If two TSMs are of the same type, then: $CD = DC$.

Truncation operation

$$\varrho(z) = \sum_{k=0}^{\infty} \omega_k z^k \quad \longrightarrow \quad \mathrm{trunc}_N\,(\varrho(z)) \stackrel{\mathrm{def}}{=} \sum_{k=0}^{N} \omega_k z^k = \varrho_N(z)$$

Function $\varrho(z)$ generates a sequence os lower TSMs:

$$L_N, \qquad N = 1, 2, \ldots$$

or upper TSMs

$$U_N, \qquad N = 1, 2, \ldots$$

Properties:

$$\mathrm{trunc}_N\,(\gamma\lambda(z)) = \gamma\ \mathrm{trunc}_N\,(\lambda(z))$$

$$\mathrm{trunc}_N\,(\lambda(z) + \mu(z)) = \mathrm{trunc}_N\,(\lambda(z)) + \mathrm{trunc}_N\,(\mu(z))$$

$$\mathrm{trunc}_N\,(\lambda(z)\mu(z)) = \mathrm{trunc}_N(\mathrm{trunc}_N\,(\lambda(z))\ \mathrm{trunc}_N\,(\mu(z)))$$

Operations with TSMs

$$A_N = \sum_{k=0}^{N} a_k (E_1^-)^k = \lambda_N(E_1^-), \quad B_N = \sum_{k=0}^{N} b_k (E_1^-)^k = \mu_N(E_1^-),$$

$$\lambda_N(z) = \mathrm{trunc}_N\,(\lambda(z))\,, \qquad \mu_N = \mathrm{trunc}_N\,(\mu(z))$$

Addition and subtraction:

$$A_N \pm B_N \longleftrightarrow \mathrm{trunc}_N\,(\lambda(z) \pm \mu(z))$$

Multiplication by a constant:

$$\gamma A_N \longleftrightarrow \mathrm{trunc}_N\,(\gamma\lambda(z))$$

Product of TSMs:

$$A_N B_N \longleftrightarrow \mathrm{trunc}_N\,(\lambda(z)\mu(z))$$

Matrix inversion:

$$(A_N)^{-1} \longleftrightarrow \mathrm{trunc}_N\,\left(\lambda^{-1}(z)\right)$$

Left-sided R-L derivatives

$${}_aD_{t_k}^{\alpha} f(t) \approx \frac{\nabla^{\alpha} f(t_k)}{h^{\alpha}} = h^{-\alpha} \sum_{j=0}^{k} (-1)^j \binom{\alpha}{j} f_{k-j}, \qquad k = 0, 1, \ldots, N.$$

$$\begin{bmatrix} h^{-\alpha}\nabla^{\alpha} f(t_0) \\ h^{-\alpha}\nabla^{\alpha} f(t_1) \\ h^{-\alpha}\nabla^{\alpha} f(t_2) \\ \vdots \\ h^{-\alpha}\nabla^{\alpha} f(t_{N-1}) \\ h^{-\alpha}\nabla^{\alpha} f(t_N) \end{bmatrix} = \frac{1}{h^{\alpha}} \begin{bmatrix} \omega_0^{(\alpha)} & 0 & 0 & 0 & \cdots & 0 \\ \omega_1^{(\alpha)} & \omega_0^{(\alpha)} & 0 & 0 & \cdots & 0 \\ \omega_2^{(\alpha)} & \omega_1^{(\alpha)} & \omega_0^{(\alpha)} & 0 & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ldots & \ldots \\ \omega_{N-1}^{(\alpha)} & \ddots & \omega_2^{(\alpha)} & \omega_1^{(\alpha)} & \omega_0^{(\alpha)} & 0 \\ \omega_N^{(\alpha)} & \omega_{N-1}^{(\alpha)} & \ddots & \omega_2^{(\alpha)} & \omega_1^{(\alpha)} & \omega_0^{(\alpha)} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{N-1} \\ f_N \end{bmatrix}$$

$$\omega_j^{(\alpha)} = (-1)^j \binom{\alpha}{j}, \qquad j = 0, 1, \ldots, N.$$

$$B_N^{\alpha} = \frac{1}{h^{\alpha}} \begin{bmatrix} \omega_0^{(\alpha)} & 0 & 0 & 0 & \cdots & 0 \\ \omega_1^{(\alpha)} & \omega_0^{(\alpha)} & 0 & 0 & \cdots & 0 \\ \omega_2^{(\alpha)} & \omega_1^{(\alpha)} & \omega_0^{(\alpha)} & 0 & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ldots & \ldots \\ \omega_{N-1}^{(\alpha)} & \ddots & \omega_2^{(\alpha)} & \omega_1^{(\alpha)} & \omega_0^{(\alpha)} & 0 \\ \omega_N^{(\alpha)} & \omega_{N-1}^{(\alpha)} & \ddots & \omega_2^{(\alpha)} & \omega_1^{(\alpha)} & \omega_0^{(\alpha)} \end{bmatrix}$$

$$\beta_{\alpha}(z) = h^{-\alpha}(1 - z)^{\alpha}.$$

$$B_N^{\alpha} B_N^{\beta} = B_N^{\beta} B_N^{\alpha} = B_N^{\alpha+\beta},$$

$${}_aD_t^{\alpha}\left({}_aD_t^{\beta} f(t)\right) = {}_aD_t^{\beta}\left({}_aD_t^{\alpha} f(t)\right) = {}_aD_t^{\alpha+\beta} f(t),$$

$$f^{(k)}(a) = 0, \quad k = 1, 2, \ldots, r - 1,$$

$$r = \max\{n, m\}$$

Left-sided R-L integration

$$ {}_aD_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \qquad (a < t < b), $$

$$ I_N^\alpha = (B_N^\alpha)^{-1}. $$

$$ I_N^\alpha \longleftrightarrow \varphi_N(z) = \mathrm{trunc}_N\left(\beta_\alpha^{-1}(z)\right) = \mathrm{trunc}_N\left(h^\alpha (1-z)^{-\alpha}\right). $$

$$ I_N^\alpha = h^\alpha \begin{bmatrix} \omega_0^{(-\alpha)} & 0 & 0 & 0 & \cdots & 0 \\ \omega_1^{(-\alpha)} & \omega_0^{(-\alpha)} & 0 & 0 & \cdots & 0 \\ \omega_2^{(-\alpha)} & \omega_1^{(-\alpha)} & \omega_0^{(-\alpha)} & 0 & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \cdots & \cdots \\ \omega_{N-1}^{(-\alpha)} & \ddots & \omega_2^{(-\alpha)} & \omega_1^{(-\alpha)} & \omega_0^{(-\alpha)} & 0 \\ \omega_N^{(-\alpha)} & \omega_{N-1}^{(-\alpha)} & \ddots & \omega_2^{(-\alpha)} & \omega_1^{(-\alpha)} & \omega_0^{(-\alpha)} \end{bmatrix} $$

Example: Riesz kernel

$$ \frac{1}{\Gamma(1-\alpha)} \int_{-1}^{1} \frac{y(\tau)\, d\tau}{|t-\tau|^\alpha} = 1, \quad (-1 < t < 1), $$

Exact solution:

$$ y(t) = \pi^{-1}\Gamma(1-\alpha)\cos\left(\frac{\alpha\pi}{2}\right)(1-t^2)^{(\alpha-1)/2}. $$

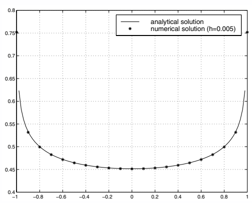
Numerical solution:

$$ {}_{-1}D_t^{-(1-\alpha)} y(t) + {}_tD_1^{-(1-\alpha)} y(t) = 1, $$

$$ (B_N^{-(1-\alpha)} + F_N^{-(1-\alpha)}) Y_N = F_N $$

Example: Riesz kernel

$\alpha = 0.8$



Example (Caputo derivatives)

$$ y^{(\alpha)}(t) + y(t) = 1, \tag{1} $$
$$ y(0) = 0, \quad y'(0) = 0, \tag{2} $$

Exact solution:

$$ y(t) = t^\alpha E_{\alpha,\alpha+1}(-t^\alpha). \tag{3} $$

Numerical solution:

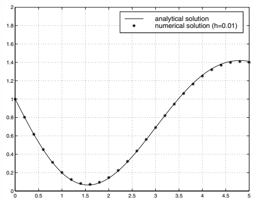
$$ \{B_{N-2}^\alpha + E_{N-2}\}\{S_{0,1}Y_N\} = S_{0,1}F_N. \tag{4} $$

and from the initial conditions we have:

$$ y_0 = y_1 = 0 $$

Example (Caputo derivatives)

$\alpha = 1.8$



Applications?

Applications (1)

New rheological models.

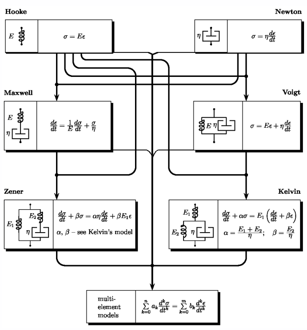
New mathematical models (laws)
of deformation of viscoelastic materials.

Applications (2)

Impact of the process history on its state.
Modelling “memory” of the process.

Fractional derivative ${}_{t_0}D_t^{\alpha} f(t)$ is used for
modelling the impact of the process history, $f(t)$
(I.e. the values of $f(t)$ for $t < t_0$)
on its state at time t_0.

Hereditary properties of materials, etc.

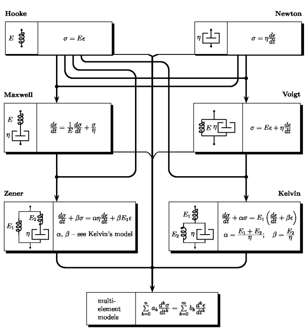


Applications (3)

Dynamical processes in fractals.

Mathematical models of *dynamical* processes
in fractals (self-similar structures or materials)
lead to FDEs, where the order(s) of the equation(s)
depends on the fractal dimension.

Porous materials, chemical reactions, diffusion,
new types of electrical circuits, physiology,
chaotic processes, econophysics, etc.



$$\sigma(t) + a\,{}_0D_t^{\alpha}\sigma(t) = b\epsilon(t) + c\,{}_0D_t^{\alpha}\epsilon(t)$$

Applications (4)

Process control.

Fractional order dynamical systems
as more adequate models of real
dynamical objects and processes.
Fractional order controllers.
Robust control.

Applications (5)

“Fractional-order” physics?

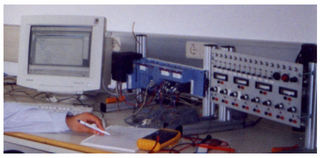
Hooke’s law: $F = kx$

Newton’s fluid: $F = kx'$ $\rightarrow$ $F(t) = kx^{(\alpha)}(t)$

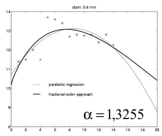
Newton’s 2nd law: $F = kx''$

Diffusion–wave equation: $\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2}$

Digital realization: PLC B & R 2005



Steel ropes in Nizna Slana (fractional)



$y_1, y_2, \dots, y_n$

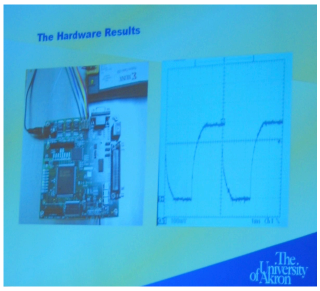
$$y(t) = \sum_{k=0}^{m-1} a_k t^k - a_m \, {}_0D_t^{-\alpha} y(t)$$

$(0 < \alpha \le m)$

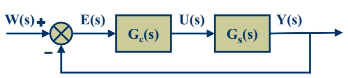
$\alpha,\ a_k\ (k = \overline{0, m}),\ m \ -?$

$${}_0D_t^{\alpha} z(t) + a_m z(t) = -a_m \sum_{k=0}^{m-1} a_k t^k; \quad z^{(k)}(0) = 0,\ (k = \overline{0, m-1})$$

Digital realization: FPGA



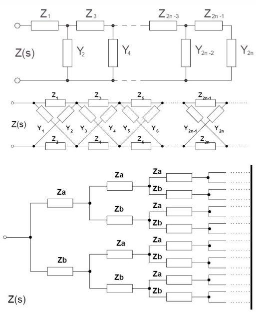
Fractional order systems and controllers



$$G_s(s) = \frac{1}{a_n s^{\beta_n} + a_{n-1} s^{\beta_{n-1}} + \ldots + a_1 s^{\beta_1} + a_0 s^{\beta_0}}$$

$$a_n D^{\beta_n} y(t) + a_{n-1} D^{\beta_{n-1}} y(t) + \ldots + a_0 D^{\beta_0} y(t) = u(t)$$

Analogue realization



Analogue realization

$$Z(s) = Z_1(s) + \cfrac{1}{Y_2(s) + \cfrac{1}{Z_3(s) + \cfrac{1}{Y_4(s) + \cfrac{1}{\cdots\cdots\cdots + \cfrac{1}{Y_{2n-2}(s) + \cfrac{1}{Z_{2n-1}(s) + \cfrac{1}{Y_{2n}(s)}}}}}}$$

$$Z(s) = \cfrac{1}{Z_1(s) + \cfrac{1}{Y_2(s) + \cfrac{1}{Z_3(s) + \cfrac{1}{\cdots\cdots\cdots + \cfrac{1}{Y_{2n-2}(s) + \cfrac{1}{Z_{2n-1}(s) + \cfrac{1}{Y_{2n}(s)}}}}}}$$

Fractor: Analogue device

Fractional Calculus Day at USU, April 19, 2005

Geometric interpretation of fractional integration: shadows on the walls

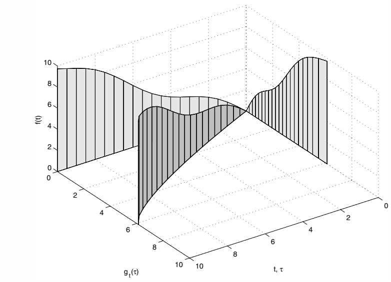
$${}_0I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(\tau)(t-\tau)^{\alpha-1}\, d\tau, \quad t \geq 0,$$

$${}_0I_t^\alpha f(t) = \int_0^t f(\tau)\, dg_t(\tau),$$

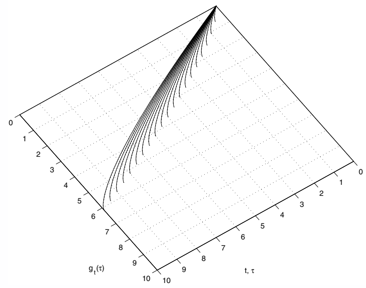
$$g_t(\tau) = \frac{1}{\Gamma(\alpha+1)}\{t^\alpha - (t-\tau)^\alpha\}.$$

For $t_1 = kt$, $\tau_1 = k\tau$ $(k > 0)$ we have:

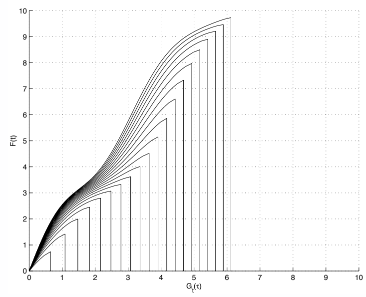
$$g_{t_1}(\tau_1) = g_{kt}(k\tau) = k^\alpha g_t(\tau).$$



"Live fence" and its shadows: ${}_0I_t^1 f(t)$ a ${}_0I_t^\alpha f(t)$, for $\alpha = 0.75$, $f(t) = t + 0.5\sin(t)$, $0 \leq t \leq 10$.



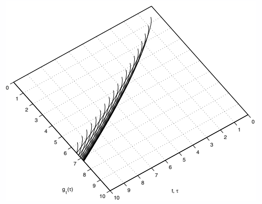
"Live fence": basis shape is changing for ${}_0I_t^\alpha f(t)$, $\alpha = 0.75$, $0 \leq t \leq 10$.



Snapshots of the changing "shadow" of changing "fence" for ${}_0I_t^\alpha f(t)$, $\alpha = 0.75$, $f(t) = t + 0.5\sin(t)$, with the time interval $\Delta t = 0.5$ between the snapshops.

Right-sided R-L integral

$$ {}_tI_0^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b f(\tau)(\tau - t)^{\alpha-1}\, d\tau, \quad t \le b, $$



${}_tI_{10}^{\alpha} f(t)$, $\alpha = 0.75$, $0 \le t \le 10$

Start
73 / 90
Back
Full screen
Close
End

How do we measure time intervals?

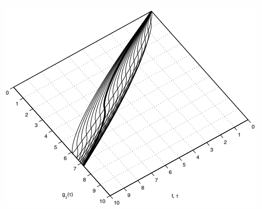
Only by observing some processes, which we consider as regularly repeated. G. Clemence (*Amer. Scientist*, vol. 40, 1952) wrote:

> "The measurement of time is essentially a process of counting. Any recurring phenomenon whatever, the occurences of which can be counted, is in fact a measure of time."

Start
76 / 90
Back
Full screen
Close
End

Riesz potential

$$ {}_0R_b^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^b f(\tau)|\tau - t|^{\alpha-1}\, d\tau, \quad 0 \le t \le b, $$



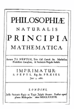
${}_0R_{10}^{\alpha} f(t)$, $\alpha = 0.75$, $0 \le t \le 10$

Start
74 / 90
Back
Full screen
Close
End

The fact that time measurement as a process of counting of repeating discrete events does not really exclude inhomogeneity of time, has been nicely mentioned by L. Carroll in *Alice's Adventures in Wonderland*:

> "…I know I have to beat time when I learn music."
>
> "Ah! That accounts for it," said the Hatter. "He [Time] won't stand beating. Now, if you only kept on good terms with him, he'd do almost anything you liked to do with the clock…"

Start
77 / 90
Back
Full screen
Close
End



What is "time"?

I. Newton (*Principia Mat.*, 1686): introduced "mathematical time":

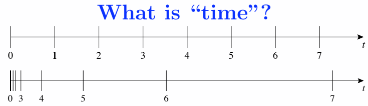
> "Absolute, true and mathematical time of itself, and from its own nature, flows equably without relation to anything external."

G. J. Whitrow (*The Natural Philosophy of Time*, 1961):

> "The outstanding mathematical achievement associated with the geometrization of time was, of course, the invention of the calculus of fluxions by Newton."
>
> "Mathematically, Newton seems to have found support for his belief in absolute time by the need, in principle, for an ideal rate-measurer."

Start
75 / 90
Back
Full screen
Close
End

What is "time"?



Computing the distance when the time is slowing down

Person N individual "seconds"	Recorded values of velocity [m/s]	Observer O absolute (cosmic) "seconds"
0	10	0
1	11	1
2	12	3
3	13	7
4	12	15
5	11	31
6	10	63
7	9	127

$$ D_N = 10\cdot 1 + 11\cdot 1 + 12\cdot 1 + 13\cdot 1 + 12\cdot 1 + 11\cdot 1 + 10\cdot 1 = 79. $$

$$ D_O = 10\cdot 1 + 11\cdot 2 + 12\cdot 4 + 13\cdot 8 + 12\cdot 16 + 11\cdot 32 + 10\cdot 64 = 1368. $$

Start
78 / 90
Back
Full screen
Close
End

Physical interpretation of Stieltjes integral (1)

Imagine a car equipped with two devices for measurements: the speedometer recording the velocity $v(\tau)$, and the clock which should show the time τ. The clock, however, shows the time incorrectly.

Suppose that the relationship between the wrong time τ, shown by the clock and considered by the driver A as the correct time, and the true time T, on the other, is described by the function $T = g(\tau)$.

Physical interpretation of Stieltjes integral (2)

Driver A's computations:

$$S_A(t) = \int_0^t v(\tau) d\tau \,.$$

Well-informed observer O's computations:

$$S_O(t) = \int_0^t v(\tau) dg(\tau).$$

This example shows that the Stieltjes integral can be interpreted as the real distance passed by a moving object, for which we have recorded correct values of speed and incorrect values of time; the relationship between the wrongly recorded time τ and the correct time T is given by a known function $T = g(\tau)$.

Physical interpretation of the Riemann-Liouville integral: "shadows of the past"

$$S_O(t) = \int_0^t v(\tau)\, dg_t(\tau) = \ {}_0I_t^{\alpha} v(t),$$

$$g_t(\tau) = \frac{1}{\Gamma(\alpha+1)} \{t^{\alpha} - (t-\tau)^{\alpha}\}.$$

The left-sided Riemann–Liouville fractional integral of the individual speed $v(\tau)$ of a moving object, for which the relationship between its individual time τ and the cosmic time T at each individual time instance t is given by the known function $T = g_t(\tau)$, represents the real distance $S_O(t)$ passed by that object.

Interpretations of the Volterra integrals

$$K * f(t) = \int_0^t f(\tau) k(t-\tau) d\tau$$

Assuming that $k(t) = K'(t)$, this integral takes the form:

$$K * f(t) = \int_0^t f(\tau) dq_t(\tau),$$

$$q_t(\tau) = K(t) - K(t-\tau).$$

The geometric and physical interpretations of the Volterra convolution integral are then similar to the suggested interpretations for fractional integrals.

What about initial conditions?

K. Diethelm, N. J. Ford, A. D. Freed, and Yu. Luchko (January 2005):

> "A typical feature of differential equations (both classical and fractional) is the need to specify additional conditions in order to produce a unique solution. For the case of Caputo FDEs, these additional conditions are just the static initial conditions . . . , which are akin to those of classical ODEs, and are therefore familiar to us. In contrast, for Riemann-Liouville FDEs, these additional conditions constitute certain fractional derivatives (and/or integrals) of the unknown solution at the initial point $x = 0$ . . . , which are functions of x. These initial conditions are not physical; furthermore, it is not clear how such quantities are to be measured from experiment, say, so that they can be appropriately assigned in an analysis."

N. Heymans and I. Podlubny:

> "Physical interpretation of initial conditions for fractional differential equations with the Riemann-Liouville fractional derivatives", *Rheologica Acta,* accepted September 1, 2005.

The end?

The end?

No! The beginning!

The beginning of a new stage

1695		1960s	You are here
static models	dynamical models	fractional order modeling	
geometry, algebra	differential and integral calculus	fractional calculus	

Instead of conclusion (1)

G.W. Scott Blair (1950):

“We may express our concepts in Newtonian terms if we find this convenient but, if we do so, we must realize that we have made a translation into a language which is foreign to the system which we are studying.”

Instead of conclusion (2)

S. Westerlund (1991):

“Expressed differently, we may say that Nature works with fractional time derivatives.”

Instead of conclusion (3)

K. Nishimoto (1989):

“The fractional calculus is the calculus of the XXI century.”

Thank you!

Web sites to start from:

http://www.tuke.sk/podlubny/
http://mechatronics.ece.usu.edu/foc/